Published by IOP Publishing for SISSA
Received: *November 5, 2008*
Accepted: *November 20, 2008*
Published: *December 17, 2008*

Non-Abelian k-vortex dynamics in $\mathcal{N} = 1^*$ theory and its gravity dual

Roberto Auzzi and S. Prem Kumar

Department of Physics, Swansea University,
Singleton Park, Swansea, SA2 8PP, U.K.
E-mail: r.auzzi@swansea.ac.uk, s.p.kumar@swansea.ac.uk

Abstract: We study magnetic flux tubes in the Higgs vacuum of the $\mathcal{N} = 1^*$ mass deformation of SU(N_c), $\mathcal{N} = 4$ SYM and its large N_c string dual, the Polchinski-Strassler geometry. Choosing equal masses for the three adjoint chiral multiplets, for all N_c we identify a "colour-flavour locked" symmetry, SO(3)$_{C+F}$ which leaves the Higgs vacuum invariant. At weak coupling, we find explicit non-Abelian k-vortex solutions carrying a $\mathbb{Z}_{N_c}$-valued magnetic flux, with winding, $0 < k < N_c$. These k-strings spontaneously break SO(3)$_{C+F}$ to U(1)$_{C+F}$ resulting in an S^2 moduli space of solutions. The world-sheet sigma model is a nonsupersymmetric $\mathbb{CP}^1$ model with a theta angle $\theta_{1+1} = k(N_c - k)\theta_{3+1}$ where θ_{3+1} is the Yang-Mills vacuum angle. We find numerically that k-vortex tensions follow the Casimir scaling law $T_k \propto k(N_c - k)$ for large N_c. In the large N_c IIB string dual, the SO(3)$_{C+F}$ symmetry is manifest in the geometry interpolating between $AdS_5 \times S^5$ and the interior metric due to a single D5-brane carrying D3-brane charge. We identify candidate k-vortices as expanded probe D3-branes formed from a collection of k D-strings. The resulting k-vortex tension exhibits precise Casimir scaling, and the effective world-sheet theta angle matches the semiclassical result. S-duality maps the Higgs to the confining phase so that confining string tensions at strong 't Hooft coupling also exhibit Casimir scaling in $\mathcal{N} = 1^*$ theory in the large N_c limit.

Keywords: Confinement, Gauge-gravity correspondence, Duality in Gauge Field Theories, Solitons Monopoles and Instantons.

JHEP12(2008)077

Contents

JHEP12(2008)077

1. Introduction

Supersymmetric gauge theories, following the works of [1], have provided a large class of examples where condensation of monopoles is the mechanism for confinement of electric charges. Softly broken $\mathcal{N} = 2$ supersymmetric gauge theories confine via a magnetic version of the Abelian Higgs mechanism. In these theories the confined, heavy, coloured sources are held together by Abelian strings (Abrikosov-Nielsen-Olesen solitons [2, 3]). In contrast, in pure Yang-Mills theory with $\mathrm{SU}(N_c)$ gauge group for instance, heavy external charges are expected to be confined by chromoelectric flux tubes which annihilate in groups of N_c, which we refer to as $\mathbb{Z}_{N_c}$-strings. One example where the dynamics of $\mathbb{Z}_{N_c}$ strings may be accessed at weak coupling, is presented by the so-called $\mathcal{N} = 1^*$ theory [4 – 9] which is a mass deformation of $\mathcal{N} = 4$ theory preserving $\mathcal{N} = 1$ supersymmetry (SUSY). What makes this theory particularly interesting is that it also has a known large N_c string dual [8] which brings with it the possibility of exploring flux tube dynamics in the large N_c limit.

The $\mathcal{N} = 1^*$ theory has extremely rich infrared dynamics and beautiful phase structure, made possible in part by the Olive-Montonen electric-magnetic duality [10] (enlarged to $\mathrm{SL}(2,\mathbb{Z})$) which it inherits from its parent $\mathcal{N} = 4$ theory. For example, the theory with $\mathrm{SU}(N_c)$ gauge group has a large number of vacua with a mass gap, each of which is realized in a distinct phase. The action of $\mathrm{SL}(2,\mathbb{Z})$ exchanges and permutes these vacua. The vacuum in the Higgs phase, where the gauge group is broken completely, is mapped by S-duality (inversion of the gauge coupling) onto the confining vacuum where the gauge group is classically unbroken and the theory confines in the infrared (IR). The $\mathbb{Z}_{N_c}$ chromoelectric flux tubes in the confined phase at strong gauge coupling, get directly mapped to magnetic flux tubes in the Higgs vacuum at weak coupling. At weak coupling, the Higgs vacuum is semiclassical and hence the physics of the associated flux tubes is accessible. The study of these for general N_c, and particularly their large N_c gravity duals , will be the subject of this paper.

In recent years, certain special flux tubes at weak coupling have been encountered in gauge theories (with and without SUSY) with $\mathrm{U}(N_c)$ gauge group and N_f flavours in the fundamental representation [11, 12] and extensively studied therein [13 – 15]. The crucial feature of all these strings at weak coupling is the presence of orientational moduli associated with rotations within a colour-flavour locked symmetry. We will refer to these as "non-Abelian" flux tubes. The interested reader can find reviews in [16 – 19]. In the context of $\mathcal{N} = 1^*$ theory with SU(2) gauge group, non-Abelian vortices in the Higgs vacuum were constructed and studied first in [20].

The basic example of the non-Abelian strings is in the context of $\mathcal{N} = 2$, $\mathrm{U}(N_c)$ gauge theory with N_f flavours and $N_f = N_c = N$ and a Fayet-Iliopolous term. In this case there is an $\mathrm{SU}(N)_{C+F}$ symmetry which is left unbroken by the vacuum; the vortex soliton breaks this symmetry to $(\mathrm{SU}(N-1) \times \mathrm{U}(1))_{C+F}$. The vortex internal space is then parameterized by

$$\mathcal{M}^{\mathcal{N}=2} = \frac{\mathrm{SU}(N)_{C+F}}{(\mathrm{SU}(N-1) \times \mathrm{U}(1))_{C+F}} = \mathbb{CP}^{N-1} \,.$$

In this paper we will study a similar "colour-flavour locked" symmetry that appears

JHEP12(2008)077

in the Higgs vacuum of $\mathcal{N}=1^*$ theory with SU(N_c) gauge group. When the masses of the three adjoint $\mathcal{N}=1$ chiral multiplets in the theory are chosen to be equal, an SO(3) subgroup of the original global SO(6)$_R$ symmetry of $\mathcal{N}=4$ theory is left unbroken. The VEVs of the scalar fields in this phase are proportional to N_c dimensional SU(2) generators. This fact allows to find a specific combination of the global SO(3) and colour generators, that are left unbroken by the VEVs of the adjoint scalars. We denote this combined colour-flavour symmetry as SO(3)$_{C+F}$.

Since all fields in the theory are in the adjoint representation of the gauge group, the topologically stable flux tubes are classified by a $\mathbb{Z}_{N_c}$ quantum number $k = 1, 2, \ldots N_c - 1$.[1] In the first part of this paper we find a general ansatz for the k-vortex solution, generalizing the $N_c = 2$, $k = 1$ case studied in [20]. The ansatz is given in explicit form for $N_c = 3, 4$ and a natural algorithm for higher rank gauge groups presents itself. Since the equations of motion are not analytically tractable, a numerical solution of the vortex profile functions is necessary. We were able to perform the numerical computations for k-vortices in theories with $2 \leq N_c \leq 6$.

The k-vortex solution breaks the SO(3)$_{C+F}$ symmetry to U(1)$_{C+F}$, so that the vortex internal moduli space (for every k) is parameterized by

$$\mathcal{M}^{\mathcal{N}=1^*} = \frac{\mathrm{SO}(3)_{C+F}}{\mathrm{U}(1)_{C+F}} = \mathbb{CP}^1 \,.$$

Acting on a given k-string solution with the broken symmetry generators rotates the orientation of the non-Abelian magnetic flux within the colour space. A crucial difference between the vortices in $\mathcal{N}=1^*$ theory and those in theories with $\mathcal{N}=2$ SUSY, is that the latter are BPS solutions. With SO(3) symmetric masses, the $\mathcal{N}=1^*$ vortices are far from BPS[2] and have no fermionic "super-orientational" zero modes.

The low-energy effective theory for the fluctuations of the light modes on the k-string is determined by performing an adiabatic, world-sheet dependent colour-flavour locked rotation. This excites the internal, orientational zero mode degrees of freedom localised on the vortex. The resulting action is that of a sigma model with S^2 as target space and the following Lagrangian,

$$S_{1+1} = \int dz\, dt \, \left(B_{N_c,k} (\partial_s \vec{n})^2 - \frac{\theta^{N_c,k}_{1+1}}{8\pi} \epsilon^{sr} \epsilon_{abc} n^a \partial_s n^b \partial_r n^c \right) , \tag{1.1}$$

where $\vec{n}$ is a position vector on the unit sphere. This is an effective theory with a UV cutoff determined by the vortex thickness. Importantly, the effective theory is an asymptotically free quantum theory and its IR dynamics depends strongly on the vacuum theta angle [14, 23 – 26]. Therfore, while the four dimensional gauge theory is semiclassical, the vortex theory is highly quantum and becomes strongly interacting. The classical value of the

[1] Flux tubes at weak coupling with $\mathbb{Z}_{N_c}$ quantum numbers were also studied in numerous papers (see [21, 22] for an incomplete list).

[2] There is a limiting regime of mass parameters (two masses equal, and the third being relatively small) where the $\mathcal{N}=1^*$ theory can be viewed as softly broken $\mathcal{N}=2^*$ theory, but we will not be particularly interested in this limit. Abelian vortices in softly broken $\mathcal{N}=2^*$ theory are BPS.

JHEP12(2008)077

sigma model coupling $B_{N_c,k}$ can be determined in terms of the Yang-Mills coupling $g^2_{\rm YM}$, for all k and N_c, by a non-trivial numerical calculation involving the vortex profile functions. The classical sigma model coupling constant turns out to be weak for weak gauge coupling. On the other hand, the effective 2-dimensional θ angle can be computed analytically in general, to yield a simple, but very interesting result,

$$\theta^{N_c,k}_{1+1} = k(N_c - k)\theta_{3+1}\,, \tag{1.2}$$

where θ_{3+1} is the vacuum angle of the four dimensional gauge theory. This relation is significant for two reasons. First, it satisfies the basic requirement that the Higgs vacuum should be invariant under shifts of θ_{3+1} by multiples of 2π. Second, whenever $\theta_{1+1} = \pi$, the world-sheet theory is integrable and flows to a conformal fixed point with massless SO(3) doublets as the only excitations. For all other values of θ_{1+1} the two dimensional theory develops a mass gap and its only excitations are triplets of SO(3) which may be viewed as confined meson-like states made up of doublets. This in turn implies that there exist various special values for θ_{3+1}, determined by (1.2) for every k, at which different k-vortex theories flow to an interacting conformal fixed point with central charge $c = 1$.

Since we find the explicit k-vortex solutions, albeit numerically, we are in a position to ask how their tensions scale with N_c. This is a question that has attracted considerable interest in recent years, from various perspectives [9, 27 – 30] for gauge theories with a $\mathbb{Z}_{N_c}$ symmetry. We perform a numerical analysis of the semiclassical k-string tensions and their ratios for $N_c = 4, 5, 6$. We find that as N_c is increased, the results are extremely well approximated by a Casimir scaling law with an accuracy better than 0.1%. Although we do not yet have an understanding of the physics behind this result, we are able to confirm that Casimir scaling of the tensions becomes precise in the large N_c gravity dual. At this point it is worth emphasizing that S-duality maps these Higgs phase results at $g^2_{\rm YM} \ll 1$ to the confining vacuum at $g^2_{\rm YM} \gg 1$.

The second part of our paper is devoted to a study of k-strings in the Higgs vacuum in the large N_c, Type IIB string dual obtained by Polchinski and Strassler [8]. The supergravity background which is dual to the Higgs vacuum, becomes applicable when $N_c \to \infty$ and $N_c/g^2_{\rm YM} \gg 1$. Since this also includes the regime of weak gauge coupling, we cannot expect supergravity to be valid in the entire geometry as a weakly coupled regime would correspond to large curvatures in the string dual. This also occurs in the Higgs vacuum, where the dual background interpolates between $AdS_5 \times S^5$ asymptotics and a deep interior portion generated by a D5-brane wrapped on a flux supported two-sphere. The D5-brane which carries N_c units of D3-brane charge, makes an appearance due to the Myers effect [31] resulting from the $\mathcal{N} = 1^*$ deformation. In the crossover region, near the D5-brane, the geometry becomes strongly curved and we expect large string corrections. Despite this we can certainly look for candidate probe brane configurations that are expected to be dual to the k-vortices of the Higgs vacuum in the large N_c limit. By S-duality, the picture in the Higgs vacuum is exchanged with the confining vacuum at strong 't Hooft coupling: $N_c/g^2_{\rm YM} \to g^2_{\rm YM}N_c \gg 1$ which is the usual condition for the validity of supergravity.

With all the above caveats in mind, we look for our candidate probe branes in the dual geometry. The $\mathrm{SO}(3)_{C+F}$ is obvious in the geometry as the sphere wrapped by the D5-

JHEP12(2008)077

brane has an SO(3) isometry in the limit of equal masses for the adjoint chiral multiplets. The $k = 1$ vortex is naturally a probe D1-brane in the Higgs vacuum. In the brane picture, the D1-brane binds to the D5-brane which has a world-volume B-field endowing the 5-brane with D3 charge. This bound state corresponds to a magnetic flux tube. In the gravity picture, the probe D1-brane sits at a radial position near the D5-brane. Despite the possibility of stringy corrections to the background, we use the probe Dirac-Born-Infeld action and the Chern-Simons terms to obtain the effective Lagragian in eq. (1.1), with

$$B_{N_c,1} = \frac{\pi N_c}{g^2_{\rm YM}}, \qquad \theta^{N_c,1}_{1+1} = N_c \theta_{3+1} \,. \tag{1.3}$$

The value of $\theta^{N_c,1}_{1+1}$ which is found in the dual is consistent with eq. (1.2) for large N_c. The tension of this configuration can also be computed (as originally done in [8]) and yields $T_{k=1} = 2\pi m^2 N_c / g^2_{\rm YM}$.

In order to model the k-string with $k \sim \mathcal{O}(N_c)$, motivated by the Myers dielectric effect on a collection of k D-strings in the Higgs vacuum, we use a D3-brane with the topology of $\mathbb{R}^{1,1} \times S^2$, with k units of flux in the compact directions. Crucially, the primary contribution to the tension of this D3-brane is a disc stretching inside the D5-sphere, a picture that we find to be consistent with the baryon vertex in $\mathcal{N} = 1^*$ theory. From the D3-brane picture we find that the vortex tension follows the Casimir scaling law

$$T_k = 2\pi \frac{m^2}{g^2_{\rm YM}} k(N_c - k) \,. \tag{1.4}$$

reproducing precisely the semiclassical field theory result which was determined numerically. Most remarkably the Chern-Simons terms of the probe brane also compute the theta angle on the k-vortex worldsheet, exactly matching eq. (1.2), the weak coupling gauge theory result.

The agreement between the gravity dual and semiclassical gauge theory physics is surprising and clearly needs an explanation. An important aspect of the probe brane results from the string dual is that the physical quantities that agree with the gauge theory - the k-string tension and the worldsheet theta angle - do not appear to receive significant contributions from the strongly curved parts of the geometry. The D3-brane k-string tension arises mainly from a disc-like portion that effectively sees a flat geometry inside the D5-sphere, while the theta angle originates in the Chern-Simons term which is insensitive to the metric. Therefore, we believe that the above picture and results are robust.

The paper is organized as follows. In section 2 we review certain aspects of the $\mathcal{N} = 1^*$ field theory. In section 3 we present solitonic solutions for the k-strings and present the results of numerical analysis. In section 4 we derive the vortex world-sheet effective action from a direct calculation. Section 5 deals with the probe brane calculation in the Polchinski-Strassler background for the Higgs vacuum. Section 6 briefly summarizes our conclusions. Some details of the interior geometry in the Polchinski-Strassler background are presented in an appendix.

JHEP12(2008)077

2. The field theory setting

In this section we cover some of the basic facts regarding the $\mathcal{N} = 1^*$ field theory with SU(N_c) gauge group. We pay particular attention to the theory in the Higgs phase, and for a specific choice of the mass deformation parameters. The physics in this vacuum is related directly via S-duality to the confining phase of the theory.

2.1 The $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM

We begin by reviewing the field content and the microscopic Lagrangian of the $\mathcal{N} = 1^*$ theory. In the language of $\mathcal{N} = 1$ supersymmetry, the $\mathcal{N} = 1^*$ theory contains an $\mathcal{N} = 1$ vector multiplet W_α and three chiral multiplets (Φ_1, Φ_2, Φ_3), transforming in the adjoint representation of the gauge group which we take to be SU(N_c). The theory is obtained by a relevant, mass deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The superpotential of $\mathcal{N} = 4$ SYM reads,

$$\mathcal{W} = \frac{1}{g^2_{\rm YM}} \, {\rm Tr}([\Phi_1, \Phi_2]\Phi_3). \tag{2.1}$$

The superpotential can be deformed by adding $\mathcal{N} = 1$ SUSY preserving mass terms for the adjoint matter fields,

$$\Delta\mathcal{W} = \frac{1}{g^2_{\rm YM}} \sum_{i=1}^{3} \frac{1}{2} m_i {\rm Tr}(\Phi_i^2). \tag{2.2}$$

This is a relevant deformation and the resulting theory exhibits nontrivial dynamics in the infrared, resulting in a rich phase structure. In the UV however, the theory flows to $\mathcal{N} = 4$ SYM, with the gauge coupling remaining a freely adjustable parameter. Thus the $\mathcal{N} = 1^*$ theory has, in addition to three complex mass parameters, a dimensionless, tunable complexified gauge coupling

$$\tau = \frac{4\pi i}{g^2_{\rm YM}} + \frac{\theta_{3+1}}{2\pi}. \tag{2.3}$$

In Euclidean space, the bosonic part of the action is,

$$S^{\rm E}_{\rm b} = \int d^4x \, \left[\frac{1}{g^2_{\rm YM}} \, \left(\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{j=1}^{3} {\rm Tr}|D_\mu \Phi_j|^2 + V_D + V_F \right) + \frac{i\,\theta_{3+1}}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \right] , \tag{2.4}$$

where we have used the same symbol Φ_j, for the chiral superfields as for their lowest (scalar) components. We define the SU(N_c) generators T^a, $(a = 1, 2, \ldots N_c^2 - 1)$, (with $F_{\mu\nu} = T_a F^a_{\mu\nu}$), with the usual normalization ${\rm Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$, while the gauge covariant derivative is

$$D_\mu \Phi_k = \partial_\mu \Phi_k - i[A_\mu, \Phi_k]. \tag{2.5}$$

The scalar potential is the sum of V_F and V_D, the F and D-term contributions respectively:

$$V_F = {\rm Tr}\left(w_1 . w_1^\dagger + w_2 . w_2^\dagger + w_3 . w_3^\dagger \right) , \qquad w_i = \epsilon_{ijk} \Phi_j \Phi_k + m_i \Phi_i \, , \tag{2.6}$$

and

$$V_D = -\frac{1}{4} {\rm Tr}\left([\Phi_1^\dagger, \Phi_1] + [\Phi_2^\dagger, \Phi_2] + [\Phi_3^\dagger, \Phi_3] \right)^2 . \tag{2.7}$$

In this paper we will be mainly interested in the case where the masses of the three adjoint chiral multiplets are equal:

$$m_1 = m_2 = m_3 = m. \tag{2.8}$$

With this choice, the superpotential term $\Delta\mathcal{W}$ breaks the $\mathrm{SO}(6)_R$ global symmetry of the $\mathcal{N} = 4$ theory to an SO(3) subgroup under which the complex chiral multiplets (Φ_1, Φ_2, Φ_3), transform as a triplet. In the $\mathcal{N} = 1^*$ theory, this SO(3) acts as an ordinary global symmetry, and not as an R-symmetry.

2.2 Higgs and confining vacua

The mass deformation above results in a large set of vacuum configurations determined by the F-flatness conditions (modulo complex gauge transformations),

$$\Phi_i = -\frac{1}{m}\,\epsilon_{ijk}\,\Phi_j\Phi_k. \tag{2.9}$$

As is well-known [5, 6], the solutions to these equations may be classified in terms of all N_c-dimensional representations the SU(2) algebra. Each such classical ground state then splits into a certain number quantum vacua depending on the non-Abelian gauge symmetry subgroup left unbroken by the classical solution. The quantum ground states are in one to one correspondence with all possible phases of $\mathrm{SU}(N_c)$ gauge theory with adjoint matter, in four dimensions.

Of particular interest are the Higgs and confining vacua which correspond to the N_c dimensional ireducible representation and the trivial representation, respectively. The VEVs of the adjoint scalars in the Higgs vacuum are proportional to the generators of the irreducible SU(2) representation with dimension N_c,

$$\Phi_l = i\,m\,J_l\,, \qquad (l = 1, 2, 3). \tag{2.10}$$

For generic N_c, the SU(2) representation is labelled by $j = \frac{N_c-1}{2}$ with J_3 chosen to be the usual diagonal matrix

$$J_3 = \mathrm{diag}(j, j-1, \ldots, -j)\,; \qquad j = \frac{N_c-1}{2}. \tag{2.11}$$

The only non-zero elements of the matrices J_1, J_2 are off-diagonal, given by

$$\begin{aligned}(J_1)_{a,\,a+1} &= (J_1)_{a+1,\,a} = \frac{\sqrt{a(N_c-a)}}{2}\,, \qquad a = 1, 2, \ldots N_c - 1 \\ (J_2)_{a,\,a+1} &= -(J_2)_{a+1,\,a} = -i\frac{\sqrt{a(N_c-a)}}{2}\,.\end{aligned} \tag{2.12}$$

The usual relation between the generators of the SU(2) algebra and the quadratic Casimir then follows,

$$J_1^2 + J_2^2 + J_3^2 = j(j+1)\mathbf{1} = \frac{(N_c^2-1)}{4}\mathbf{1}\,. \tag{2.13}$$

This relation leads to a natural association of the Higgs vacuum of $\mathcal{N} = 1^*$ theory with fuzzy sphere configurations of D3 branes in the string theory dual [8].

The results we deduce below for magnetic flux tubes in the Higgs vacuum, will have a direct bearing on the tension of the chromoelectric flux tubes in the confining vacuum. This is because the $SL(2,\mathbb{Z})$ electric-magnetic duality of the parent $\mathcal{N}=4$ theory permutes different IR phases of the $\mathcal{N}=1^*$ theory [4–6]. In particular, the Higgs and confing vacua are exchanged under S-duality: $\tau \to -1/\tau$.

2.3 Colour-flavour locking

The VEVs of the adjoint scalars in the Higgs vacuum break the SO(3) global symmetry and the $SU(N_c)$ gauge symmetry. However, it is always possible to find a combined global colour-flavour rotation which is unbroken [20]. This combined $SO(3)_{C+F}$ global symmetry subgroup can be understood as follows. Any global SO(3) rotation of the triplet (Φ_1, Φ_2, Φ_3) can be undone by a global colour transformation whose generators are chosen to be proportional to the VEVs of the adjoint scalars i.e., the N_c dimensional SU(2) generators. More explicitly, we first rotate the triplet (Φ_1, Φ_2, Φ_3) with the flavour matrix $U_F = \exp(T_j a_j)$, where T_j are the following SO(3) generators,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

This transformation acts in the flavour space as

$$\vec{\Phi} \to U_F \, \vec{\Phi} \, . \tag{2.15}$$

Then let us introduce the global colour matrix $W_C = \exp(i J_l a_l)$, acting as:

$$\Phi_i \to W_C \, \Phi_i \, W_C^\dagger, \tag{2.16}$$

where J_l are the N_c dimensional representations of SU(2) generators.

A combination of the above flavour and colour rotations are unbroken by the scalar VEVs. The existence of this $SO(3)_{C+F}$ symmetry allows the determination of the world-sheet sigma model of vortices (magnetic flux tubes) in the Higgs vacuum, as we see below.

2.4 Higgs vacuum spectrum

The perturbative spectrum in the Higgs vacuum for $N_c = 2$ is given by an $SO(3)_{C+F}$ triplet of massive vector $\mathcal{N}=1$ multiplets with mass $\sqrt{2}m$, one chiral multiplet with mass m and 5 chiral multiplets with mass $2m$. For general N_c the perturbative spectrum has been computed in [34]. The result is the following: there are always 3 massive vectors multiplets with mass $\sqrt{2}m$ and one chiral multiplet[3] with mass m. In addition, for every $k = 2, \ldots, N_c - 1$ there are $4k$ massive chiral multiplets with mass km, $2k+1$ massive vectors with mass $\sqrt{k(k+1)}m$ and lastly, a set of $2N_c + 1$ chiral multiplets with mass $N_c m$. For every N_c all these particles fit in representations of $SO(3)_{C+F}$.

[3] In [34], the spectrum of the $U(N_c)$ gauge theory was determined, which differs slightly from the $SU(N_c)$ theory discussed here. In particular, for $U(N_c)$, there are three additional chiral multiplets with mass m and one massless vector multiplet.

JHEP12(2008)077

A beautiful feature of the Higgs vacuum of the $\mathcal{N} = 1^*$ theory is that in the large N_c limit it provides a deconstruction of a six dimensional theory compactified on a sphere. In particular, as discussed in [34], the perturbative spectrum of the U(N_c), $\mathcal{N} = 1^*$ theory, is identical to the spectrum of the Maldacena-Nuñez twisted compactification of the $\mathcal{N} = (1, 1)$ six dimensional U(1) gauge theory on a two-dimensional sphere. This interpretation is a direct consequence of the association of the Higgs vacuum with a fuzzy sphere configuration [35] as described above.

3. The $\mathbb{Z}_{N_c}$ vortex as a soliton

3.1 General discussion

Since the $\mathcal{N} = 1^*$ theory has only fields transforming in the adjoint representation of the gauge group, the Lagrangian is invariant under transformations in the center $\mathbb{Z}_{N_c}$ of SU(N_c). In the Higgs vacuum, magnetic charges valued in $\mathbb{Z}_{N_c} = \pi_1 [\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}]$ are confined by magnetic flux tubes, also carrying a $\mathbb{Z}_{N_c}$ charge. Since the fluxes are defined modulo N_c, they annihilate in groups of N_c. At weak coupling $g_{\rm YM} \ll 1$, the physics in the Higgs vacuum is semiclassical and the magnetic flux tubes should be understood as ordinary non-Abelian vortex string solutions of the classical equations of motion.

In this section we introduce an ansatz for solitonic k-strings in the Higgs vacuum of the $\mathcal{N} = 1^*$ theory with gauge group SU(N_c) in the semiclassical limit $g_{\rm YM} \ll 1$. We will write the ansatz explicitly for $N_c = 2, 3, 4$. The resulting vortices carry magnetic flux $k = 1, 2, \ldots, N_c - 1$, defined modulo N_c. Since they annihilate in groups of N_c, $N_c = 4$ is the minimal gauge group for which a non-trivial $k = 2$ string appears. There is also a $k = 2$ vortex for $N_c = 3$, but is essentially equivalent to the $k = 1$ vortex under the transformation $k \to N_c - k$.

The generalization of our ansatz to general k and N_c is straightforward, but explicit calculations with these ansätze get quite complicated. For generic N_c and k, in order to solve the equations of motion of the gauge theory, we need to introduce an ansatz which depends on $3(N_c - 1)$ independent profile functions for the vortex. We have performed explicit numerical computations for $2 \leq N_c \leq 6$ and generic k.

The classical equations of motion for the bosonic fields read

$$\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = \frac{i}{2} \sum_{l=1}^{3} \Big([D^\nu \Phi_l, \Phi_l^\dagger] + [D^\nu \Phi_l^\dagger, \Phi_l] \Big) \,, \tag{3.1}$$

$$D^\mu D_\mu \Phi_i = (m\, w_i^\dagger - \epsilon_{ijl} [w_j^\dagger, \Phi_l^\dagger]) + \frac{\partial V_D}{\partial \Phi_i} \,. \tag{3.2}$$

Below we list explicit vortex solutions which satisfy

$$\Phi_i = -\Phi_i^\dagger \,, \qquad i = 1, 2, 3. \tag{3.3}$$

so that the D-term contribution to the potential is identically zero when evaluated on the solution, and the resulting equations of motion are somewhat simpler. For general N_c, there are $N_c - 1$ distinct topological sectors labelled by an integer k with $0 < k < N_c$.

The vortex configurations have the adjoint scalars Φ_i approaching, at infinity, a gauge transform of their VEVs in the Higgs vacuum. In particular, certain matrix elements of the adjoint scalars undergo a 2π phase rotation upon winding once around the vortex. This phase rotation corresponds to a gauge transformation (at infinity) which is single-valued in $\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}$. In our ansätze below, the solutions with winding number $k = 1$ will have the scalars winding at infinity, effectively generated by

$$Y_1 = \frac{1}{N_c}\,\mathrm{Diag}(1, \ldots, 1, -(N_c - 1)), \tag{3.4}$$

resulting in a chromomagnetic flux proportional to Y_1. Thus the flux picks out a specific direction in colour-flavour space and the associated string is truly non-Abelian.

In these solutions, Φ_3 is chosen to have no azimuthal variation, whilst both Φ_1 and Φ_2 have nontrivial angular dependence, away from the vortex core. In particular, as the azimuthal angle φ varies from 0 to 2π, the components $(\Phi_{1,2})_{N_c-1,N_c}$ wind with a phase $e^{i\varphi}$, while $(\Phi_{1,2})_{N_c,N_c-1}$ wind with the opposite phase $e^{-i\varphi}$.

For generic winding $1 < k < N_c$ the flux carried by the corresponding k-vortex is proportional to

$$Y_k = \mathrm{Diag}\left(\underbrace{\frac{k}{N_c}, \ldots, \frac{k}{N_c}}_{N_c - k\,\text{elements}},\ -\frac{N_c - k}{N_c}, \ldots, -\frac{N_c - k}{N_c}\right), \tag{3.5}$$

and away from the vortex core, the adjoint scalars behave as

$$\Phi_{1,2}(r, \varphi) = e^{iY_k\varphi}\ \Phi_{1,2}(r, \varphi = 0)\ e^{-iY_k\varphi} \tag{3.6}$$

while Φ_3 has only a radial dependence. Under the effect of this rotation, the components $(\Phi_{1,2})_{N_c-k,N_c-k+1}$ and $(\Phi_{1,2})_{N_c-k+1,N_c-k}$ wind around the vortex with phase $e^{i\varphi}$ and $e^{-i\varphi}$ respectively. We believe that these are the solutions of lowest tension in each topological sector k, as each field winds at infinity exactly once. The solutions also display an obvious vortex/anti-vortex symmetry, which is evident under the replacement $k \to N_c - k$.

The explicit vortex solutions will break the $\mathrm{SO}(3)_{C+F}$ global symmetry. However, they are invariant under the action of a U(1) subgroup corresponding to global rotations acting on (Φ_1, Φ_2). The action of the broken global symmetry generators then leads to a $\mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathbb{CP}^1$ moduli space of solutions for generic (N_c, k).

3.2 $N_c = 2$

For SU(2) gauge group, the vortex solutions were first found in [20]. Here we rederive their result for completeness,

$$\Phi_1 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}, \tag{3.7}$$

$$\Phi_3 = \frac{im}{2}\kappa_1(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

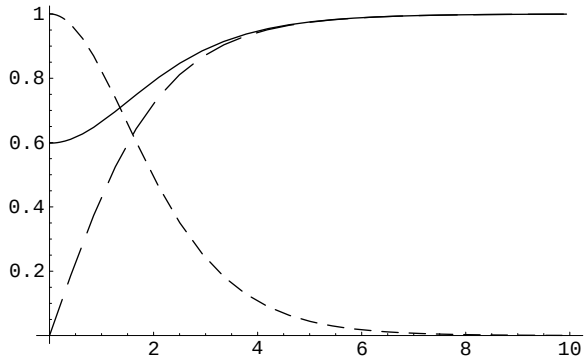

Figure 1: The vortex profile functions for $N_c = 2$: κ_1 (solid), ψ_1 (long dashes), f (short dashes).

where ψ_1 and κ_1 are profile functions to be determined by the equations of motion. Both approach unity as $r \to \infty$, in order to match up with the Higgs VEVs. Near the origin ψ_1 vanishes so that the solution is smooth at $r = 0$. It is obvious that this configuration will be invariant under a combination of an SO(2) flavour rotation acting on the pair (Φ_1, Φ_2) and a global colour rotation generated by σ_3. Hence the full $\mathrm{SO}(3)_{C+F}$ is broken to $\mathrm{U}(1)_{C+F}$.

The gauge field is solved by a typical vortex form

$$A_x = \frac{-y}{r^2}(1 - f(r))Y, \qquad A_y = \frac{x}{r^2}(1 - f(r))Y, \quad Y = \frac{1}{2}\sigma_3. \tag{3.8}$$

This picks out a direction in the colour space and results in a magnetic flux also proportional to Y,

$$F_{xy} = -\frac{f'(r)}{r}\, Y. \tag{3.9}$$

The ansatz which is axisymmetric under rotations about the z-axis, can be used to evaluate the action functional per unit length. This yields the vortex tension functional

$$T = 2\pi \int r dr \left(\frac{f'^2}{2r^2} + \frac{m^2 \kappa_1'^2}{2} + m^2 \psi_1'^2 + \frac{m^2 \psi_1^2 f^2}{r^2} + \frac{m^4}{2}((\kappa_1 - \psi_1^2)^2 + 2\psi_1^2(\kappa_1 - 1)^2) \right). \tag{3.10}$$

It is easily checked that the equations of motion for the profile functions that follow from varying this tension functional are the same as those following from (3.1) and (3.2). The profile functions are thus determined by solving,

$$f'' - \frac{f'}{r} - f\psi_1^2 2m^2 = 0, \tag{3.11}$$

$$\psi_1'' + \frac{\psi_1'}{r} - \frac{\psi_1 f^2}{r^2} = m^2 \psi_1 (\psi_1^2 + \kappa_1^2 - 3\kappa_1 + 1), \tag{3.12}$$

$$\kappa_1'' + \frac{\kappa_1'}{r} = m^2 (2\psi_1^2 \kappa_1 - 3\psi_1^2 + \kappa_1). \tag{3.13}$$

The above equations can be solved numerically and the results are plotted in figure 1. We learn from the solution that ψ_1 grows from zero at the core of the vortex to unity at

infinity. At the same time, κ_1 remains non-zero at the string core whilst approaching 1 asymptotically. Hence, in the core $\Phi_3 \neq 0$, $\Phi_1 = \Phi_2 = 0$, so that there is a Coulomb-like phase, shielded by a crossover region which eventually merges with the Higgs phase vacuum at infinity. The profile function $f(r)$ shows that the magnetic field $\sim f'(r)/r$ is non-zero in the Coulomb-like phase, concentrated in a neighbourhood of the origin, while vanishing in the asymptotic Higgs vacuum.

3.3 $N_c = 3$

Having understood the structure of the $\mathbb{Z}_2$ string for SU(2), we can now apply our general non-Abelian string ansatz described in section 3.1, to higher rank gauge groups. For SU(3), and for $k = 1$ our general ansatz takes the form,

$$\Phi_1 = \frac{im}{\sqrt{2}} \begin{pmatrix} 0 & \psi_1 & 0 \\ \psi_1 & 0 & \psi_2 e^{i\varphi} \\ 0 & \psi_2 e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{\sqrt{2}} \begin{pmatrix} 0 & -i\psi_1 & 0 \\ i\psi_1 & 0 & -i\psi_2 e^{i\varphi} \\ 0 & i\psi_2 e^{-i\varphi} & 0 \end{pmatrix}, \quad (3.14)$$

$$\Phi_3 = im \begin{pmatrix} \kappa_1 - \kappa_2/2 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & -\kappa_1 - \kappa_2/2 \end{pmatrix}.$$

The forms of Φ_1 and Φ_2 are both motivated by their Higgs phase VEVs, imJ_1 and imJ_2 respectively. At infinity, the profile functions ψ_1 and ψ_2 approach unity, while ψ_2 vanishes at the origin so that the solution remains smooth. As we go around the origin, Φ_1 and Φ_2 undergo a phase rotation generated by

$$Y_1 = \frac{1}{3}\mathrm{Diag}(1, 1, -2). \quad (3.15)$$

The magnetic flux for the solution turns out to be proportional to Y_1, which satisfies

$$\exp(2\pi i Y_1) = \mathrm{Diag}(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}). \quad (3.16)$$

The gauge field is modified slightly from the SU(2) case,

$$A_x = \frac{-y}{r^2}\left((1 - f(r))Y_1 + g(r)\lambda\right), \qquad A_y = \frac{x}{r^2}\left((1 - f(r))Y_1 + g(r)\lambda\right), \quad (3.17)$$

where $\lambda = \frac{1}{2}\mathrm{Diag}(1, -1, 0)$ and $g(r)$ is a new profile function. The non-Abelian magnetic field then is

$$F_{xy} = -\frac{f'(r)}{r} Y_1 + \frac{g'(r)}{r} \lambda. \quad (3.18)$$

As in the SU(2) example, we can evaluate the energy per unit length for the ansatz to obtain the tension functional which can be varied to yield the equations of motion,

$$T = 2\pi \int r\, dr \Bigg(\frac{2f'^2}{3r^2} + \frac{g'^2}{2r^2} + \frac{1}{2}m^2(4\kappa_1'^2 + 3\kappa_2'^2) + 2m^2(\psi_1'^2 + \psi_2'^2) + \quad (3.19)$$
$$+ \frac{1}{2r^2}m^2\big(4g^2\psi_1^2 + (g - 2f)^2\psi_2^2\big) + \frac{1}{2}m^4\Bigg(\frac{1}{2}(2\kappa_1 - \kappa_2 - 2\psi_1^2)^2 + \frac{1}{2}(2\kappa_1 + \kappa_2 - 2\psi_2^2)^2$$
$$2(\kappa_2 + \psi_1^2 - \psi_2^2)^2 + (2 - 2\kappa_1 + 3\kappa_2)^2\psi_1^2 + (2 - 2\kappa_1 - 3\kappa_2)^2\Bigg)\Bigg).$$

Once again the equations of motion following from this functional are consistent with the equations of motion of the full theory, eqs. (3.1) and (3.2).

For SU(3) gauge group there is also a $k = 2$ vortex. However this is follows from a $k \to N_c - k$ replacement in our $k = 1$ solution. In other words, the $k = 2$ solution will be the identical to the above, with the opposite flux (winding).

3.4 $N_c = 4$

The $\mathbb{Z}_{N_c}$ string solution for SU(4) gauge group is particularly interesting, as this is the first instance where we encounter a non-trivial multi-vortex solution, i.e. with winding $k > 1$. We only need to consider the cases with $k = 1, 2$ (the $k = 3, 4$ vortices are identical to $k = 1, 2$ respectively with negative winding).

$k = 1$ solution. The ansatz follows the general pattern described earlier,

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 & 0 \\ 0 & 2\psi_2 & 0 & \sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & \sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix}, \tag{3.20}$$

$$\Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 & 0 \\ 0 & i2\psi_2 & 0 & -i\sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & i\sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

The following expression is used for the gauge field:

$$A_x = \frac{-y}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \qquad A_y = \frac{x}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \tag{3.21}$$

where

$$Y_1 = \frac{1}{4}\mathrm{Diag}(1,1,1,-3). \tag{3.22}$$

which yields $\exp(2\pi i Y_1) = \mathrm{Diag}(e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2})$. The non-Abelian flux carried by the vortex is proportional to Y_1. The g_ℓ's are functions of r vanishing both at $r = 0$ and at $r \to \infty$, and λ_ℓ are a basis of diagonal matrices with satisfying,

$$\mathrm{Tr} Y_1 \lambda_\ell = 0, \qquad \mathrm{Tr}\lambda_i\lambda_\ell = \frac{1}{2}\delta_{i\ell}. \tag{3.23}$$

We choose

$$\lambda_1 = \frac{1}{\sqrt{12}}\mathrm{Diag}(1,1,-2,0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(1,-1,0,0). \tag{3.24}$$

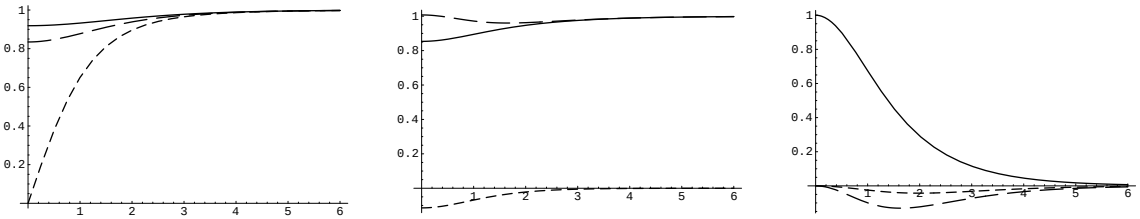

Figure 2: The vortex profile for $N_c = 4$. Left: ψ_1 (solid), ψ_2 (long dashes), ψ_3 (short dashes). Center: κ_1 (solid), κ_2 (long dashes), κ_3 (short dashes). Right: f (solid), g_1 (long dashes), g_2 (short dashes).

The string profile can then can be found by the minimization of the energy functional. We do not write the explicit form as it is quite lengthy. The numerical solutions to the resulting equations of motion are shown in figure 2.

Since ψ_3 vanishes at the origin and all the diagonal elements of Φ_3 remain non-zero at $r = 0$, we infer that at the core of the vortex solution, a U(1) subgroup of the gauge symmetry is unbroken and theory is in a Coulomb phase in that region.

***k* = 2 solution.** We now turn to the $k = 2$ vortex solution. The relevant configuration for the scalars is now obtained by applying an SU(4) rotation to the Higgs vacuum VEVs, generated by

$$Y_2 = \frac{1}{2}\,\mathrm{Diag}(1, 1, -1, -1) \tag{3.25}$$

with $\exp(2\pi i Y_2) = \mathrm{Diag}(-1, -1, -1, -1)$. The chromomagnetic flux is also proportional to Y_2. The explicit ansatz is then,

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 e^{i\varphi} & 0 \\ 0 & 2\psi_2 e^{-i\varphi} & 0 & \sqrt{3}\psi_3 \\ 0 & 0 & \sqrt{3}\psi_3 & 0 \end{pmatrix}, \tag{3.26}$$

$$\Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 e^{i\varphi} & 0 \\ 0 & i2\psi_2 e^{-i\varphi} & 0 & -i\sqrt{3}\psi_3 \\ 0 & 0 & i\sqrt{3}\psi_3 & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

The gauge fields are still given by eq. 3.21, with Y_1 replaced by Y_2 and

$$\lambda_1 = \frac{1}{2}\mathrm{Diag}(1, -1, 0, 0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(0, 0, 1, -1). \tag{3.27}$$

As before the vortex profiles can be found numerically and the results are shown in figure 3.

JHEP12(2008)077

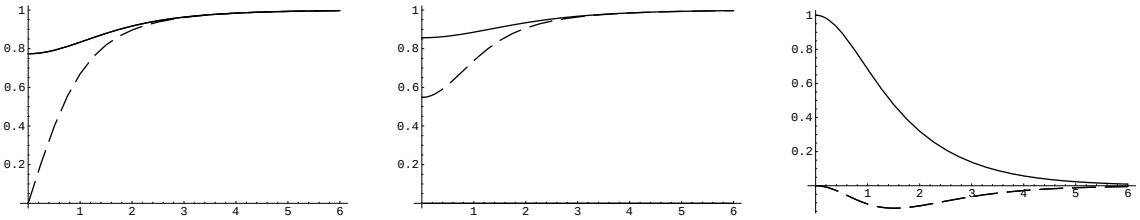

Figure 3: Profiles for $N_c = 4$ and $k = 2$. Left: $\psi_1 = \psi_3$ (solid), ψ_2 (long dashes). Center: κ_1 (solid), κ_2 (long dashes), $\kappa_3 = 0$. Left: f (solid), $g_1 = g_2$ (long dashes).

This solution provides a confirmation of the general picture of these non-Abelian vortices in the Higgs vacuum of $\mathcal{N} = 1^*$ theory. They all have an unbroken U(1) gauge group at their core, while approaching a totally Higgsed phase in the exterior. This is, of course, consistent with the premise that the Higgs vacuum and its excitations should have a semiclassical description. Another general feature is that the non-Abelian strings break the $SO(3)_{C+F}$ symmetry group to a U(1) subgroup. The moduli space of $\mathbb{Z}_{N_c}$ string solutions is therefore isomorphic to $\mathbb{CP}^1 \simeq SU(2)_{C+F}/U(1)$ for all k and N_c. We will also confirm this feature of the theory in its large N_c string dual.

The generalization of the vortex ansatz to arbitrary N_c proceeds in a straightforward fashion and requires introducing $3(N_c - 1)$ profile functions $(\psi_i, \kappa_i, f, g_i)$. In the absence of any obvious analytical simplifications, we will not pursue this direction further in this paper.

3.5 k-string tensions

The study of non-Abelian k-string tensions is a topic of great interest and is particularly so in the present context. The non-Abelian vortices of the Higgs vacuum at weak coupling $g_{\rm YM} \ll 1$ are mapped by S-duality of $\mathcal{N} = 1^*$ theory to confining strings at strong coupling $g_{\rm YM} \gg 1$. With the explicit ansätze at hand for general N_c and k, we can compute their tensions, albeit numerically. We will then compare these results with the known tensions in a different parametric regime for $\mathcal{N} = 1^*$ theory wherein the vortex strings are almost BPS. It should be pointed out that when $m_1 = m_2 = m_3 = m$, the strings are *far from* BPS. Nevertheless we will see that the numerical values of the tensions approach the BPS values as N_c is increased.

The tension of semiclassical non-Abelian strings in the Higgs vacuum of $\mathcal{N} = 1^*$ theory has been discussed in [20, 22], in the limit

$$m_1 = m_2 = m, \quad \text{and} \quad m_3 \ll m. \tag{3.28}$$

In this limit the vortex becomes an almost BPS object, due to the fact that in the limit $m_3/m \to 0$, $\mathcal{N} = 2$ supersymmetry is restored. The theory may then be viewed as softly broken $\mathcal{N} = 2^*$ theory. The $\mathcal{N} = 2^*$ theory, with $m_1 = m_2 = m$ and $m_3 = 0$, is realized in the Coulomb phase due to Φ_3 obtaining a VEV. Adding a mass m_3 for Φ_3 at the appropriate point on the Coulomb branch moduli space results in complete Higgsing of the theory due to electric degrees of freedom becoming light and condensing. The profile

N_c	2	3	4	5	6
$k=1$	0.894	0.926	0.943	0.954	0.961
$k=2$			0.944	0.954	0.962
$k=3$					0.962

Table 1: Values of $T_{N_c,k}/T^{\rm BPS}_{N_c,k}$ for $2 \leq N_c \leq 6$ and different k.

functions of the vortices in this limit are simpler, because it is consistent to take the profiles $\kappa_j(r)$ (equivalently, Φ_3) as constant. For a BPS vortex the tension is exactly proportional to the field condensates:

$$T^{BPS}_{N_c,k} = 2\pi \, \frac{m m_3}{g^2_{\rm YM}} \, k(N_c - k). \tag{3.29}$$

This behaviour is the so-called 'Casimir scaling' of k-string tensions.

The case we have focussed attention on this paper is far from the BPS limit with,

$$m_1 = m_2 = m_3 = m. \tag{3.30}$$

Using our ansatz above we have numerically evaluated the vortex tension functional $T_{N_c,k}$ for $2 \leq N_c \leq 6$ and the results are in table 1. In this table we have presented the ratio of the tensions for the $SO(3)_{C+F}$ symmetric theory to the BPS formula (3.29) extrapolated to $m_3 = m$.

The main conclusion that we can draw from this numerical data is that for large N_c the k-string tension $T_{N_c,k}$ in the theory with $m_3 = m$ quickly approaches the BPS tension formula given by eq. (3.29). There does not appear to be an obvious explanation for this result. We also note also that for fixed N_c the ratios in the table are, to a very good approximation, independent of k.

The numerical results for string tension ratios $T_{N_c,k+1}/T_{N_c,k}$ are also rather striking. For $N_c = 4$ we find the following numerical result,

$$\frac{T_{N_c=4,\,k=2}}{T_{N_c=4,\,k=1}} = 1.334 \tag{3.31}$$

while the prediction from Casimir scaling is 4/3. For $N_c = 5$ we find

$$\frac{T_{N_c=5,\,k=2}}{T_{N_c=5,\,k=1}} = 1.501 \tag{3.32}$$

while the Casimir scaling prediction is 3/2. Finally, for $N_c = 6$:

$$\frac{T_{N_c,\,k=2}}{T_{N_c=6,\,k=1}} = 1.6008, \qquad \frac{T_{N_c=6,\,k=3}}{T_{N_c=6,\,k=1}} = 1.801, \tag{3.33}$$

while the Casimir scaling values are 8/5 and 9/5.

The numerical results above are striking in that the tension is not a BPS protected quantity, so the accuracy of the Casimir scaling law is better than what we could expect. The Casimir scaling law is only exact in the limit $m_3 \ll m$, but evidently it is still an

extremely good approximation also for $m = m_3$ for the cases $N_c = 4, 5, 6$ which have been studied numerically. This suggests that in the large N_c theory, the k-string tensions likely obey a Casimir scaling law in the $\mathcal{N} = 1^*$ theory. This can be best understood by investigating the known large-N_c string dual of the $\mathcal{N} = 1^*$ theory [8], which we will do in section 5.

4. Effective world-sheet theory

In this section we round off our field theoretic analysis with the construction of the (classical) world-sheet theory of k-strings in the Higgs vacuum. For SU(2) gauge group this was already done in [20]. Below we will extend this to SU(N_c) gauge group and general k. We will also present a new ingredient, namely the effect on the worldsheet sigma model, of a non-zero θ_{3+1} angle in the $\mathcal{N} = 1^*$ Yang-Mills theory.

The general class of vortex solutions presented above have the property that they are invariant only under a U(1) subgroup of the colour-flavour locked $\mathrm{SO}(3)_{C+F}$ transformations. This unbroken U(1) is a rotation acting on the pair (Φ_1, Φ_2) which can be undone by a gauge transformation. The moduli space of inequivalent solutions is thus $\mathbb{CP}^1 \simeq \mathrm{SU}(2)_{C+F}/\mathrm{U}(1)$. The associated moduli correspond to the orientational modes of the magnetic flux in the string solution.

The low-lying excitations of the worldsheet theory of the vortex will involve, apart from translational zero modes for the center of mass, the adiabatic dynamics of the orientational zero modes. For all N_c and k we see that this is a nonsupersymmetric sigma model (as in the examples discussed in refs. [20, 32]) with target space $\mathbb{CP}^1$, along with a theta angle that is related in a special way to the four dimensional Yang-Mills theta angle. The absence of supersymmetry makes the present situation different from BPS non-Abelian vortex strings in $\mathcal{N} = 2$ SQCD [11, 15, 12, 14] and also different from the Heterotic vortex string discussed in refs. [36].

4.1 Kinetic term

Let us consider a vortex oriented along the z axis. In order to obtain the effective world-sheet theory of the orientational zero modes, we introduce an adiabatic $\mathrm{SO}(3)_{C+F}$ rotation which depends on the world-sheet coordinates (z, t) of the vortex string. Doing so will turn these moduli (the global rotation parameters) into worldsheet dependent fields. It is best to perform these steps in singular gauge, i.e. where the scalar fields have no winding at infinity and the flux is concentrated near the origin as in [12, 14].

Upon a (worldsheet dependent) colour-flavour locked rotation, the triplet of scalar fields transform in the following way:

$$\vec{\Phi} \to U_F(z,t) \cdot \Big(W_C(z,t)\, \vec{\Phi}\, W_C^\dagger(z,t) \Big)\,, \tag{4.1}$$

where the matrix U_F acts in the three dimensional flavour space (2.15) and W_C is a colour transformation (2.16) generated by the N_c dimensional representation of SU(2) generators. The gauge fields transform as:

$$A_{x,y} \to W_C\, A_{x,y}\, W_C^\dagger\,. \tag{4.2}$$

A transformation dependent on z and t, will of course also generate components of A_μ along the world-sheet coordinates, as evident from the ordinary gauge transformation $A_s \to W_C\, A_s\, W_C^\dagger + i W_C\, \partial_s W_C^\dagger$. To be consistent therefore, the full vortex solution will need to be modified and the radial dependence of the new components of the gauge field have to be solved for. We will, using a natural axisymmetric ansatz for all A_μ, obtain the worldsheet effective theory. This can be done along the lines of [32]: A_s are choosen in gauge space in such a way that they are perpendicular to the $A_{x,y}$ and to the derivative $\partial_s(A_{x,y})$. This is

Let us discuss, for simplicity, the $N_c = 2$ $k = 1$ case (already studied in [20]) and subsequently generalize. In singular gauge

$$\Phi_1 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{4.3}$$

$$\Phi_3 = \frac{im}{2}\kappa_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_x = \frac{y}{r^2}f(r)\frac{\sigma_3}{2}, \qquad A_y = \frac{-x}{r^2}f(r)\frac{\sigma_3}{2}\,. \tag{4.4}$$

Since the solution is symmetric under $\mathrm{U}(1)_{C+F}$ rotations around the Φ_3 axis in flavour space, to generate a new solution we need to act on it with one of the broken generators. If this action is chosen to be global, we simply obtain a new solution with the the same tension as the old one, but with the non-Abelian flux pointing in a different direction in colour+flavour space. Let us therefore consider a (z,t) dependent rotation around the Φ_2 axis without loss of generality. We will use the following worldsheet dependent colour and flavour transformations:

$$W_C\, \sigma_3\, W_C^\dagger = \vec{n}(z,t)\cdot\vec{\sigma}\,. \tag{4.5}$$

Then the following ansätze can be used for the gauge field components A_s with $s = z, t$:

$$A_s = -\,(\vec{n}\times\partial_s\vec{n})^a\,\frac{\sigma^a}{2}\rho(r)\,. \tag{4.6}$$

The gauge orientations for A_s are dictated by the requirement that they be orthogonal in colour space, to both $A_{x,y}$ and $\partial_s A_{x,y}$ (after the world-sheet dependent transformation). We substitute the expression for the gauge fields (4.6) along with the world-sheet dependendent transformation (4.2) of the solution (4.4), into the action and obtain

$$S_{1+1} = \int dz\, dt B_{2,1}(\partial_s\vec{n})^2\,, \tag{4.7}$$

where

$$B_{2,1} = \frac{1}{g^2_{\mathrm{YM}}}\int_0^\infty dr\, 2\pi r\left(\frac{\rho'^2}{2} + \frac{(\rho-1)^2 f^2}{2r^2} + \frac{m^2}{2}(2(\kappa_1-\psi_1)^2(1-\rho) + (\kappa_1^2+\psi_1^2)\rho^2)\right)\,. \tag{4.8}$$

It is worth noting that this kinetic term for the world-sheet moduli is generated only by gauge kinetic terms depending on $F_{s\mu}$ and scalar kinetic terms $\sim |D_s\Phi_{1,2}|^2$ of the four dimensional gauge theory. All other terms including the scalar potential of the gauge theory are invariant under the combined colour-flavour rotations. In order for the sigma model coupling to be finite, we need to impose the boundary conditions

$$\rho(0) = 1\,, \qquad \rho(r\to\infty) = 0. \tag{4.9}$$

N_c	2	3	4	5	6
$\frac{g^2_{\rm YM}}{2\pi} B_{N,1}$	0.390	1.181	2.343	3.867	5.847
$\frac{g^2_{\rm YM}}{2\pi} B_{N,2}$			2.696	4.344	6.710
$\frac{g^2_{\rm YM}}{2\pi} B_{N,3}$					6.888

Table 2: Some numerical results for the classical kinetic term for the k-vortex. For $N_c = 2$ there is agreement with the value computed in ref. [20].

The Euler-Lagrange equations for $\rho(r)$ and other profile functions can be solved numerically to yield the kinetic term for the world-sheet moduli fields. The result, shown in table 2 for SU(2): $B_{2,1} = 0.39\,(2\pi/g^2_{\rm YM})$ matches with that of [20].

The generalization of the above arguments to the SU(N_c) case, for each of the stable k-vortices, is actually straightforward. Now the relevant colour transformations are generated by the N_c dimensional representation of SU(2) generators. To get the normalization of the kinetic term of the resulting $\mathbb{CP}^1$ model, we again consider just a rotation around the Φ_2 axis so that,

$$W_C\, J_3\, W_C^\dagger = \vec{n}(z,t)\cdot\vec{J}\,. \tag{4.10}$$

Then the following ansatz can be used for the gauge fields along the worldsheet

$$A_s = -\,(\vec{n}\times\partial_s\vec{n})^a\; J^a\rho(r)\,. \tag{4.11}$$

If we insert these expressions into the gauge theory action, the following term in the vortex effective theory can be found,

$$S_{1+1} = \int dz\,dt\;\left(B_{N_c,k}(\partial_s\vec{n})^2\right). \tag{4.12}$$

The general formula for $B_{N_c,k}$ is complicated and we can only evaluate it on a case by case basis, numerically.

For example, for $N_c = 3$, $k = 1$ the explicit expression is,

$$\begin{aligned} B_{3,1} = \frac{1}{g^2_{\rm YM}}\int dr 2\pi r\Big(&2\rho'^2 + \frac{\left(4f^2-4gf+5g^2\right)(\rho-1)^2}{4r^2} + \frac{1}{2}m^2\big(4\left(\rho^2-2\rho+2\right)\kappa_1^2 + \\ &+8(\rho-1)\left(\psi_1+\psi_2\right)\kappa_1 + 3\left(3\rho^2-6\rho+4\right)\kappa_2^2 - 12(\rho-1)\kappa_2\left(\psi_1-\psi_2\right) \\ &+2\left(-4\psi_1\psi_2(\rho-1)^2 + \left(3\rho^2-6\rho+4\right)\psi_1^2 + \left(3\rho^2-6\rho+4\right)\psi_2^2\right)\big)\Big)\,. \end{aligned} \tag{4.13}$$

The coefficient is then determined numerically by variation of the action functional with the boundary conditions $\rho(r=0)=1$ and $\rho(r\to\infty)=0$. In table 2 are shown the numerical values for $B_{N_c,k}$ for $2 \le N_c \le 6$.

4.2 World-sheet theta angle

Whenever the Yang-Mills theta angle θ_{3+1} is non-vanishing, a new ingredient appears in the vortex world-sheet theory. The Yang-Mills theta angle feeds into the world-sheet theory

as a topological term for the $\mathbb{CP}^1$ sigma model. The coefficient of this topological term, the world-sheet theta angle denoted as θ_{1+1}, plays a crucial role in the ensuing world-sheet dynamics. In particular, the IR dynamics is strongly theta-dependent [32, 24 – 26].

We begin by demonstrating the mechanism of generation of the world-sheet theta angle for SU(2) gauge group. In this case the steps involved and the result are rather similar to [32]. The relevant terms can be obtained by a colour-flavour transformation that depends on both z and t. It will be sufficient to consider the following (z,t) dependent colour-flavour rotation of the vortex fields,

$$U_F = \exp(J_2\alpha(z)).\exp(J_1\beta(t))\,, \qquad W_C = \exp\left(i\frac{\sigma_2}{2}\alpha(z)\right).\exp\left(i\frac{\sigma_1}{2}\beta(t)\right)\,. \tag{4.14}$$

The time and space dependent rotation will generate gauge field components A_z and A_t. These will have to be chosen normal, in colour space, to $A_{x,y}$ and their derivatives. Using the appropriate ansatz, $A_s = -(\vec{n}\times\partial_s\vec{n}).(\vec{\sigma}/2)\rho(r)$, we obtain

$$\begin{aligned} A_z &= \left(\frac{\sigma_1}{4}\sin\alpha\sin 2\beta + \frac{\sigma_2}{2}\cos^2\beta - \frac{\sigma_3}{4}\cos\alpha\sin 2\beta\right)\rho(r)\,\alpha'(z)\,, \\ A_t &= \left(\frac{\sigma_1}{2}\cos\alpha + \frac{\sigma_3}{2}\sin\alpha\right)\rho(r)\,\dot{\beta}(t)\,. \end{aligned} \tag{4.15}$$

Introducing the new world-sheet variations into the space-time action action, the theta dependent topological term in the Yang-Mills action then gives rise to a topological term on the world-sheet of the vortex

$$S^\theta_{1+1} = \frac{\theta_{3+1}}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{\theta_{3+1}}{16\pi^2}\int dz\,dt\, C\alpha'(z)\dot{\beta}(t)\cos\beta\,, \tag{4.16}$$

where

$$C = \int_0^\infty 2\pi\,dr\,r\,\frac{1}{2r}\frac{d}{dr}\left(\rho^2 - 2\rho)f\right) = \pi\,. \tag{4.17}$$

Note that C is obtained by integrating a total derivative and only depends on the values of the profile functions at zero and infinity, namely $\rho(0) = f(0) = 1$ and $\rho(\infty) = f(\infty) = 0$. Written more covariantly, this leads to the following interaction in the vortex effective action,

$$\mathcal{L}^\theta_{1+1} = -\frac{\theta_{3+1}}{8\pi}\epsilon^{s,r}\,\epsilon^{abc}\,n^a\partial_s n^b\partial_r n^c \qquad s,r = (t,z). \tag{4.18}$$

This is very similar to the case discussed in ref. [32] and relates the theta angle of the $\mathbb{CP}^1$ model to the Yang-Mills theta angle as

$$\theta_{1+1} = \theta_{3+1} \qquad \text{for} \quad \mathrm{SU}(2). \tag{4.19}$$

The general result for arbitrary N_c and k is more illuminating than the SU(2) theory. In particular, in the general case the world-sheet theta angle is not equal to the space-time theta angle; the two are related and this relation depends both on N_c and k. We may consider the most general colour-flavour rotated gauge field configurations in eqs. (4.10) and (4.11) and evaluate the topological term on these to produce

$$\mathcal{L}^\theta_{1+1} = -\,C_{N_c,k}\,\frac{\theta_{3+1}}{8\pi^2}\,\epsilon^{sr}\,\epsilon^{abc}\,n^a\partial_s n^b\partial_r n^c. \tag{4.20}$$

The proportionality constant $C_{N_c,k}$ is again given by the integral of a total derivative,

$$C_{(N_c,k)} = \int_0^\infty 2\pi\, dr \frac{d}{dr} \left\{ (\rho^2 - 2\rho)\, (f(r) \mathrm{Tr}(Y_{N_c,k} J_3)) \right\} = \pi\, k(N_c - k). \tag{4.21}$$

This means that for the k-vortex, the theta angle of the world-sheet sigma model is determined by θ_{3+1} as

$$\theta_{1+1} = k(N_c - k)\theta_{3+1}. \tag{4.22}$$

So the long-wavelength fluctuations of the world-sheet theory of the $\mathbb{Z}_{N_c}$ flux tube carrying k units of magnetic flux, are governed by the effective action

$$S_{1+1} = \int dz\, dt \left(B_{N_c,k} (\partial_s \vec{n})^2 - k(N_c - k) \frac{\theta_{3+1}}{8\pi} \epsilon^{sr}\, \epsilon^{abc}\, n^a \partial_s n^b \partial_r n^c \right), \tag{4.23}$$

The effective theta angle is an integer multiple of the four dimensional one and is thus guaranteed to respect the invariance of the Higgs vacuum under $\theta_{3+1} \to \theta_{3+1} + 2\pi$.

4.2.1 Dynamics on the vortex world-sheet

We have seen that the effective long-wavelength dynamics of the k-vortices in the Higgs vacuum of $\mathcal{N} = 1^*$ theory with SU(N_c) gauge group (and with three equal masses), is given by a $\mathbb{CP}_1$ model for all N_c and k. The four dimensional theory being $\mathcal{N} = 1$ supersymmetric, the vortices are non-BPS and the effective world-sheet theory is non-supersymmetric. Thus there are no fermionic super-orientational zero modes. The resulting world-sheet dynamics is different from that of BPS vortex strings in $\mathcal{N} = 2$ SQCD [11, 12] for example.

It is well-known that the value of the theta angle has a strong effect on the IR dynamics of the $\mathbb{CP}^1$ model [23 – 26, 38, 32]. First of all the $\mathbb{CP}^1$ model is asymptotically free and so is a strongly coupled theory. This is interesting: the four dimensional field theory is weakly coupled, but the dynamics on the vortex is highly quantum. When $\theta_{1+1} = 0$ and $\theta_{1+1} = \pi$, the model is exactly solvable. Specifically, the spectrum at $\theta_{1+1} = 0$ is known to consist of a single massive SO(3) triplet with an exact S-matrix [24] and the theory has a mass gap. This picture continues to be valid for generic non-zero values of θ_{1+1}. When θ_{1+1} hits π, however, something drastic happens. The theory has massless excitations and flows to a $c = 1$ conformal fixed point [37, 39, 25] described by the SU(2) Wess-Zumino-Witten model at level $k = 1$. The spectrum now consists of massless SU(2) doublets. The picture, therefore, is that at generic θ_{1+1}, the doublets are confined and bound into meson-like excitations, transforming as a triplet of SO(3). The singlet state, not having a conserved quantum number, is unstable. It is possible to analyze the spectrum in the vicinity of $\theta_{1+1} = \pi$ [32, 38] and can be interpreted as consisting of "kink-anti-kink" bound states. The string tension between these kinks and anti-kinks (the SU(2) doublets) vanishes as the vacuum angle approaches π.

The existence of the non-trivial dynamics near $\theta_{1+1} = \pi$ begs the question: how is this reflected in the physics of the four dimensional gauge theory? The situation is particularly intriguing, since nothing obviously drastic happens in the gauge theory when $\theta_{3+1} = \pi/k(N_c - k)$. This merits deeper study, but one obvious possibility is that this concerns the spectrum of confined monopole-dyon states in the Higgs phase. The doublets

(kinks) are likely to be the bound states of monopoles with the vortex. These monopole-dyon states exist as massive 't Hooft-Polyakov monopoles in the $\mathcal{N} = 2^*$ theory in the Coulomb phase. As θ_{3+1} is dialled, the spectrum of these massive states undergoes a rearrangement and can lead to level crossing between certain mutually non-local states (e.g. the (0,1) monopole and the (1,1) dyon for SU(2)). It is possible that the special values of θ_{3+1} may be the points at which such massive, mutually non-local states become degenerate. If both these states happen to get confined upon breaking the supersymmetry to $\mathcal{N} = 1^*$, then they can appear bound to the magnetic flux tubes. The appearance of such mutually non-local states simultaneously on the world-sheet, may drive the sigma model to an interacting fixed point. We are merely speculating at this stage, but clearly the issue deserves deeper study.

5. The vortex in the string dual

The string theory dual of $\mathcal{N} = 1^*$ theory was constructed by Polchinski and Strassler [8] by considering an appropriate deformation of Type IIB string theory on $AdS_5 \times S^5$ background. The undeformed $AdS_5 \times S^5$ background with N_c units of Ramond-Ramond five form flux is dual to the large N_c limit of SU(N_c), $\mathcal{N} = 4$ SYM. The relation between gauge theory and string theory parameters is as follows. The string coupling g_s and the radius of curvature of AdS space are related to the gauge coupling and the 't Hooft coupling respectively as

$$4\pi g_s = g^2_{\rm YM}\,, \qquad \frac{R_{\rm AdS}}{\sqrt{\alpha'}} = (4\pi g_s N_c)^{1/4} \gg 1\,, \qquad C_0 = \frac{\theta_{3+1}}{2\pi} \tag{5.1}$$

where C_0 is the Type IIB RR scalar.

The $\mathcal{N} = 1^*$ mass deformation of the $\mathcal{N} = 4$ theory is achieved by switching on a non-normalizable mode for the three-form flux $G_3 = F_3 - \tau H_3$, with $\tau = i/g_s + C_0/2\pi$, the unperturbed Type IIB coupling. The Polchinski-Strassler dual geometry was obtained by treating the G_3 flux as a perturbation and solving the Type IIB supergravity equations of motion to linear order in this perturbation. The rich infrared physics of $\mathcal{N} = 1^*$ theory was captured in the string dual using two central ingredients: the Myers dielectric effect [31] and the action of SL$(2,\mathbb{Z})$ duality on the vacua of the theory.

The classical description of the $\mathcal{N} = 1^*$ vacua [8], shows that the scalars get noncommuting expectation values describing fuzzy sphere configurations [35]. In the language of D-branes, this means that that N_c D3-branes on which the parent $\mathcal{N} = 4$ theory lives, acquire non-commuting positions, transverse to their worldvolume. These transverse positions trace out fuzzy S^2's and the configuration can be reinterpreted as 5-branes wrapped on concentric flux supported two-cycles carrying N_c units of D3-brane charge. The full large N_c, IIB string dual background interpolates between the "near-shell" geometry generated by the multiple 5-branes and the asymptotically AdS solution towards the boundary of the space. Different $\mathcal{N} = 1^*$ vacua, with the theory realized in different phases, are obtained by the action of the IIB SL$(2,\mathbb{Z})$ transformations on a given fivebrane configuration.

5.1 Polchinski-Strassler Higgs vacuum

The Polchinski-Strassler description of each $\mathcal{N} = 1^*$ vacuum consists of an asymptotically AdS geometry with a G_3 flux turned on, matching onto an interior geometry generated by (c, d) 5-branes. For instance, classical vacua preserving an SU(p) gauge symmetry, where p is a divisor of N_c are described by p coincident D5 branes carrying a net D3 charge, provided

$$\frac{q}{p}\frac{1}{g_s} \gg 1\,, \qquad q = \frac{N_c}{p}. \tag{5.2}$$

When p and q are such that $pg_s/q \gg 1$, the vacuum with SU(p) gauge symmetry is described by q NS5 branes.

The Higgs vacuum is thus described by a single D5 brane carrying net D3 charge, when

$$\frac{N_c}{g_s} \gg 1 \tag{5.3}$$

while the confining vacuum is described by a single NS5 brane when

$$N_c\, g_s \gg 1\,. \tag{5.4}$$

The former is an extremely weak condition in the large N_c limit, while the latter is the usual condition for the gauge theory to be strongly coupled. The two vacua and these two conditions for the validity of the Polchinski-Strassler supergravity description in the "far-from-shell" region, are exchanged under the S-duality, $g_s \leftrightarrow 1/g_s$.

In the Higgs vacuum, with $m_1 = m_2 = m_3 = m$, the metric in the interior matches onto the geometry generated by a D5-brane wrapped on an S^2 carrying N_c units of D3-charge. The D3-brane worldvolume coordinates are x^μ, $(\mu = 0, 1, 2, 3)$ wherein the field theory lives. The six transverse directions are denoted as

$$w^i = x^{7,8,9}\,, \qquad \text{and} \qquad y^i = x^{4,5,6}. \tag{5.5}$$

The D3 branes spread out along the w^i directions with $y^i = 0$ and the resulting D5-brane wraps a round sphere of radius

$$r_0 = \pi\alpha' m N_c. \tag{5.6}$$

The string frame metric of the Polchinski-Strassler solution [8] corresponding to the Higgs vacuum (D5-brane) with equal masses for the adjoint chiral multiplets and with $\theta_{3+1} = 0$, is

$$ds^2_{\text{string}} = Z_x^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z_y^{1/2}(dy^2 + y^2 d\Omega_y^2 + dw^2) + Z_\Omega^{1/2} w^2 d\Omega_w^2, \tag{5.7}$$

where

$$Z_x = Z_y = \frac{R^4_{\text{AdS}}}{\rho_+^2\rho_-^2}, \qquad Z_\Omega = \frac{R^4_{\text{AdS}}\,\rho_-^2}{\rho_+^2(\rho_-^2+\rho_c^2)^2}, \qquad \rho_\pm = \sqrt{y^2 + (w \pm r_0)^2}. \tag{5.8}$$

The parameters in the metric are

$$R^4_{\text{AdS}} = 4\pi g_s N_c \alpha'^2, \qquad \rho_c = (\alpha' m)\sqrt{g_s N_c \pi}\,. \tag{5.9}$$

and r_0 is as defined in (5.6). The dilaton is non-constant, approaching its asymptotic value g_s as $w^i, y^i \to \infty$, but vanishing close to the D5/D3 brane,

$$e^{2\Phi} = g_s^2 \frac{\rho_-^2}{\rho_-^2 + \rho_c^2}, \qquad C_0 = \theta_{3+1} = 0. \tag{5.10}$$

The Polchinski-Strassler (including the R-R and NS-NS potentials) background has a manifest SO(3) isometry acting on the S^2 in the w-plane. This naturally gets identified with the colour-flavour locked $SO(3)_{C+F}$ symmetry of the Higgs phase of $\mathcal{N} = 1^*$ theory.

The full solution above is approximate, and is constructed by matching the $r \gg r_0$ limit with the "near-shell" solution for $r \approx r_0$. The "far-from-shell" region is well approximated by the background generated by D3-brane charge density spread out on a spherical shell and is close to a Coulomb branch configuration. In this regime the D3-brane charge density dominates over the D5 charge density. The "near-shell" regime is well described by the exact solution for a flat D5-brane with D3-brane charge [41, 42]. Here the effect of the D5-brane dominates. The flat D5 solution and the matching used by Polchinski and Strassler are reviewed in the appendix.

An important feature of the Higgs vacuum geometry is that near the D5-brane, supergravity ceases to be applicable. This is due to large transverse curvatures near the D5-brane. We will, however, adopt a pragmatic approach and use the metric for our subsequent analysis, with the aim of identifying certain aspects of the gauge theory vortex dynamics that are robustly captured by the string dual. The questions that we are interested in, will, perhaps surprisingly, turn out to be insensitive to the strongly curved parts of the geometry.

The main ingredient we will need from the region near the spherical D5 shell, is the NS-NS two-form potential (see the appendix for further details),

$$B_2 = -\frac{\alpha' \pi N_c}{1 + \rho_-^2/\rho_c^2} \sin\theta_w \, d\theta_w \wedge d\phi_w \, . \tag{5.11}$$

Near the shell there are also the non-vanishing R-R potentials, C_2 and C_4, which can be extracted from the flat D5 background discussed in appendix. The form of the R-R potentials at $\theta_{3+1} = 0$ will not be relevant for the probe branes that we will study in this section.

The Higgs vacuum solution at any non-zero theta angle for the gauge theory follows upon acting on the solution with $\theta_{3+1} = 0$ with an $SL(2,\mathbb{R})$ transformation in IIB supergravity. Under such a transformation, $\tau \to \tau + \frac{\theta_{3+1}}{2\pi}$, the 5-brane remains a D5-brane, but the two R-R potentials C_2 and C_0 are shifted as,

$$C_0 \to C_0 + \frac{\theta_{3+1}}{2\pi} \qquad C_2 \to C_2 + \frac{\theta_{3+1}}{2\pi} B_2 \, . \tag{5.12}$$

Thus at non zero θ_{3+1}, the R-R two form C_2 acquires components along the $d\theta_w \wedge d\phi_w$ direction which will be relevant for our probes.

5.2 The vortex as a D1-brane

Magnetic flux tubes in the Higgs vacuum have a natural brane interpretation as bound states of D1-branes with the D5-brane. Such D1-D5/D3 bound states involving the single

D5-brane in the Higgs vacuum are possible due to the non-vanishing B_2 potential on the 5-brane, responsible for the D3-charge. In this situation, the bound state is a semiclassical instanton of the non-commutative field theory [40] on the 5-brane.

We first review the computation of the D-string tension, dual to a $k = 1$ vortex, in the Higgs vacuum geometry, first done in [8]. The two new ingredients in our analysis will be a derivation of the world-sheet sigma model of the vortex and the effect of the Yang-Mills theta angle.

Following [8], we model a magnetic vortex as a D1-brane probe in this geometry. The DBI action for the probe D1 brane in the geometry reads,

$$S_{\rm DBI} = \frac{1}{2\pi\alpha'}\int d^2\xi \left\{ e^{-\Phi}\sqrt{(-\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}))}\right\}. \tag{5.13}$$

Let us consider a D1-brane oriented in the x_0, x_1 directions and the embedding $(\xi_0, \xi_1) = (x_0, x_1)$, so that the pullback of the metric onto the world-sheet is

$$G_{00} = -Z_x^{-1/2} + \cdots, \qquad G_{11} = Z_x^{-1/2} + \cdots \tag{5.14}$$

The dots correspond to terms involving fluctuations of the string in the transverse $\vec{y}$ and $\vec{w}$ directions, proportional to the derivatives $\partial_0(\vec{y}, \vec{w})$, $\partial_1(\vec{y}, \vec{w})$. While the fluctuations along $\vec{y}$ coordinates cannot be studied in supergravity due to large curvatures, angular fluctuations in the $\vec{w}$ directions will appear to be accessible. The part of the DBI action which does not depend on these derivatives and yields the effective tension of the D1-brane is,

$$\begin{aligned} S_{\rm DBI} &= \frac{1}{2\pi\alpha'}\int d^2x (Z_x^{-1/2} e^{-\Phi}) \\ &= \frac{1}{2\pi\alpha'}\int d^2x \frac{\sqrt{y^2+(w+r_0)^2}\sqrt{y^2+(w-r_0)^2+\rho_c^2}}{g_s\, R_{\rm AdS}^2}. \end{aligned} \tag{5.15}$$

This is minimized when,

$$w = \frac{r_0 + \sqrt{r_0^2 - 2\rho_c^2}}{2} \approx r_0 - \frac{\rho_c^2}{2r_0}, \qquad y = 0. \tag{5.16}$$

Since $\rho_c \ll r_0$, the probe D1-brane sits at a relatively small distance δw from the D5 shell,

$$\delta w = \frac{\rho_c^2}{2r_0} = \frac{g_s\alpha' m}{2}. \tag{5.17}$$

The dilaton and the functions that determine the metric at this point evaluate to

$$e^{-\Phi} \approx 2\sqrt{\frac{\pi N_c}{g_s^3}}, \qquad Z_x^{-1/2} \approx \frac{\sqrt{g_s N_c \pi}\alpha' m^2}{2}, \qquad Z_\Omega^{1/2} = \frac{1}{2}\sqrt{\frac{g_s}{(\pi N_c)^3}}\frac{1}{\alpha' m^2}. \tag{5.18}$$

The curvature of the space transverse to the D5-brane , i.e., in the radial w and $\vec{y}$ directions can be seen to be substringy at the value of w giving the location of the D1-brane. However, as we see below, this value for w corresponds to a large radius sphere in the $w^{1,2,3}$ space.

Continuing to use the metric (5.7), the tension of the D1-brane at this location is

$$T_{\rm D1} \approx \frac{2r_0\rho_c}{g_s R^2_{\rm AdS}}\,\frac{1}{2\pi\alpha'} = \frac{N_c m^2}{2g_s} = \frac{2\pi N_c m^2}{g^2_{\rm YM}}\,. \tag{5.19}$$

Remarkably, this formula matches the BPS formula (3.29) for $k=1$, in the Higgs vacuum which is only expected to work for softly broken $\mathcal{N}=2^*$ theory. This is suggestive that the vortices at large N_c (and $N_c/g_s \gg 1$) in the Higgs vacuum become BPS objects. Equivalently, the confining strings at large N_c and $g_sN_c \gg 1$ (S-dualizing the Higgs vacuum) obey the BPS tension formula. More evidence in support of this possibility was offered in [8]. This is also in agreement with our semiclassical results for large N_c.

At its equilibrium position the D1-brane is transverse to a two-sphere at $y=0$ and $w = r_0 - g_s\alpha' m/2$, which is concentric with the dielectric 5-brane sphere. At this location the radius of the transverse two-sphere is

$$Z_\Omega^{1/4} w \approx \frac{R_{\rm AdS}}{2} = \frac{1}{2}(4\pi g_s N_c)^{1/4}\sqrt{\alpha'}. \tag{5.20}$$

Clearly, this is large in string units in the supergravity limit. The D-string is pointlike on the transverse two-sphere, resulting in a $\mathbb{CP}^1$ moduli space of vortex solutions. The sphere is large in string units and we can allow for a slow, adiabatic variation of the D1-brane position on the sphere, as a function of the world-sheet coordinates. The polar coordinate of the D-string $\vec{n}_w \equiv (\theta_w, \phi_w)$, corresponds to the vortex colour-flavour zero mode.

Let us consider an arbitrary dependence of $\vec{n}_w$ on the world-sheet coordinates (x_0, x_1) and introduce this into the DBI action. Taking into account only the contribution of the pullback of the spacetime metric, the following world-volume action results,

$$S_{\rm DBI} = \int d^2x\,\frac{e^{-\Phi}}{2\pi\alpha'}\sqrt{-{\rm Det}\begin{pmatrix} -Z_x^{-1/2} + (\partial_0\vec{n}_w)^2 w^2 Z_\Omega^{1/2} & (\partial_0\vec{n}_w)\cdot(\partial_1\vec{n}_w) w^2 Z_\Omega^{1/2} \\ (\partial_0\vec{n}_w)\cdot(\partial_1\vec{n}_w) w^2 Z_\Omega^{1/2} & Z_x^{-1/2} + (\partial_1\vec{n}_w)^2 w^2 Z_\Omega^{1/2} \end{pmatrix}} \tag{5.21}$$

In this formula we have actually omitted the pullback of the B_2 field, which only contributes to a four-derivative term in the vortex world-volume action that we neglect. At the two-derivative level, we find

$$S_{\rm DBI} \approx T_{\rm D1}\int d^2x\left(1 + (\partial_s\vec{n}_w)^2\frac{w^2 Z_\Omega^{1/2} Z_x^{1/2}}{2}\right), \tag{5.22}$$

where $T_{\rm D1}$ is the tension of the $k=1$ vortex in eq. (5.19). From this we get the ceofficient of the kinetic term of the $\mathbb{CP}^1$ sigma model,

$$\mathcal{L}_{\rm kin} = \frac{N_c}{4g_s}(\partial_s\vec{n}_w)^2 = \frac{\pi N_c}{g^2_{\rm YM}}(\partial_s\vec{n}_w)^2\,. \tag{5.23}$$

While it is interesting to perform the above formal manipulations, it is not clear that the classical coupling constant of the sigma model is significant since the $\mathbb{CP}^1$ model is asymptotically free and the coupling constant will run when the sigma model is quantized.

JHEP12(2008)077

Secondly, the coupling can also get large corrections due to stringy effects from the highly curved transverse parts of the geometry. The coefficient of the topological term in the $\mathbb{CP}^1$ model, on the other hand, has special significance. We now turn to evaluating this from the D1-brane action.

5.2.1 The theta term

To complete the supergravity analysis of the $k = 1$ vortex, we will now see how the Yang-Mills theta terms feeds into the world-sheet sigma model. This feeding-in occurs through the Chern-Simons terms of the D1-brane world-volume theory,

$$S_{\rm CS} = \frac{1}{2\pi\alpha'} \int \left[\exp(2\pi\alpha' F_2 + B_2) \wedge \sum_q C_q \right] = \frac{1}{2\pi\alpha'} \int \left[C_2 + C_0\, B_2 \right] \tag{5.24}$$

We expect this term to have universal, robust features for two reasons. First, the effect of a non-zero θ_{3+1} has to be such that physics is periodic under shifts of the theta angle by 2π. This is particularly true in the Higgs vacuum which is actually invariant under such shifts, while the confining vacuum gets mapped to an oblique-confining phase under the same operation. Furthermore, the Wess-Zumino term in the D-brane action is insensitive to the background metric and may well capture the correct physics even in the supergravity approximation.

Since the probe D1-brane is located relatively very close to the D5-brane shell, we need to use the expressions for B_2, C_2 and C_0 given in (5.12), (5.11) and (A.6) for the "near-shell" region. There is a subtlety surrounding the Wess-Zumino couplings of D-branes involving the pullback of the B_2 field (such as $C_0 \wedge B_2$ in eq. (5.24)), which has been discussed in the references [43]. The upshot of this is that the contribution to Wess-Zumino terms from the pullback of B_2 have to be omitted. The term $C_0 \wedge B_2$ is an effective D(-1) 'charge' for our probe D1, arising from the pullback of the B field. In our case the contribution corresponds to a D1 world-sheet wrapping an S^2 with $|\vec{w}| \approx r_0 - \rho_c^2/2r_0$, which is a homotopically trivial cycle. From the results of [43], this term is cancelled by bulk contributions, so we have to drop it from the present calculation.

Finally then, the only relevant term is the pullback of the components of C_2 along the sphere at constant w and $y = 0$,

$$S_{\rm CS} = \frac{1}{2\pi\alpha'} \int C_2 = \frac{1}{2\pi\alpha'} \frac{\theta_{3+1}}{2\pi} \int B_2\big|_{\theta_{3+1}=0}. \tag{5.25}$$

In the large N_c supergravity limit, the magnitude of the near shell B field (5.11) at the the radial position of the D1-brane, is equal to $\alpha'\pi N_c$. The result is a two-derivative theta term for the effective $\mathbb{CP}^1$ sigma model,

$$\mathcal{L}_\theta = \frac{\theta_{1+1}}{8\pi}\, \epsilon^{sr}\, \vec{n}_w \cdot (\partial_s \vec{n}_w \times \partial_r \vec{n}_w)\ , \qquad \theta_{1+1} = N_c \theta_{3+1} \qquad (s,r) = x^{0,1}\,. \tag{5.26}$$

This is consistent (at large N_c) with our semiclassical field theory calculation done in the previous section. Note that if we were to keep the term proportional to $C_0 \wedge B_2$, we would find $\theta_{1+1} = 2N_c\theta_{3+1}$, which would be inconsistent with our semiclassical expectation.

JHEP12(2008)077

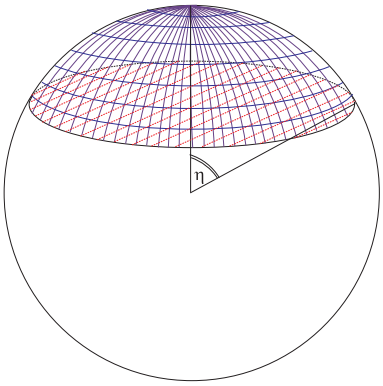

Figure 4: The minimal energy configuration for the probe D3 brane in the $\vec{w}$ space is given by the red disc and the blue "polar cap" shown in the figure. The "polar cap" part is located very close to the D5 sphere.

5.3 The k-vortex as a D3 brane

Although we were able to reproduce the sigma model of the $k = 1$ vortex using the dual geometry, the tensions and the theta terms for k-strings have a nontrivial dependence on k. It is not *a priori* clear that the Polchinski-Strassler geometry should be able to reproduce these since the IR geometry for the Higgs vacuum will receive large stringy corrections. However, as before, we will attempt to identify the appropriate D-brane configuration dual to a magnetic k-string and investigate whether this can compute the tensions and world-sheet parameters reliably.

It is now well understood in a variety of different confining backgrounds [28] that k-string tensions with k of order N_c in large N_c theories, are computed by expanded brane configurations. A collection of multiple probe F/D-strings can blow up into higher dimensional D-branes by a version of the Myers effect, wrapping topologically trivial cycles. A similar, very closely related phenomenon also occurs for Wilson loops in general tensor representations involving sources of varying N-ality in large N_c gauge theories [44 – 46]. In all these cases, the expanded brane configuration carries a net k-string charge by virtue of world-volume electric or magnetic fields.

In our large N_c dual to the Higgs vacuum, we look for candidate branes that correspond to k-vortices with

$$k \to \infty \, , N_c \to \infty \quad \text{and} \quad \frac{k}{N_c} \quad \text{fixed.} \tag{5.27}$$

The most natural object is a D3-brane with topology $\mathbb{R}^{1,1} \times S^2$, and a nonvanishing (magnetic) F_2 flux along its compact directions located near the D5 shell. Our candidate probe D3-brane is sketched in figure 4. It is located at $\vec{y} = 0$ and has the topology of an S^2 (but not the shape) in the $\vec{w}$ space. We believe this to be the correct configuration for large enough k. For finite or small k, the S^2 of the probe D3-brane will be a small, smooth, squashed sphere located at some polar angle along the D5-sphere. As k is increased, the squashed sphere grows in size with k. Most of this probe D3 sphere will want to stay near the D5 shell where the B_2 field is concentrated, in order to minimize its tension. The

brane will, however, remain blown up due to the magnetic field on its worldvolume which provides it with the requisite D-string charge. The candidate probe brane also breaks the SO(3) isometry to U(1) rotations around the w_3 axis, as we expect for the k-vortices.

5.3.1 A warm-up

As a warm-up excercise, let us compute the DBI action for a spherically symmetric configuration at constant $|\vec{w}|$, with k units of uniform flux on top of it. This will have higher tension than the D-brane in figure 4. We choose a constant world-volume magnetic field F_2 along the (θ_w, ϕ_w) directions and proportional to the volume form of S^2,

$$F_2 = \frac{k}{2} \sin\theta_w d\theta_w \wedge d\phi_w. \tag{5.28}$$

The magnetic field induces a D-string charge k for the spherical D3-brane, through its Chern-Simons coupling

$$S_{\rm CS} = \frac{1}{(2\pi)^3 \alpha'^2} \int_{S^2\times\mathbb{R}^{1,1}} 2\pi\alpha' F_2 \wedge C_2 = \frac{1}{2\pi\alpha'}\, k \int_{\mathbb{R}^{1,1}} C_2. \tag{5.29}$$

The tension for the spherical D-brane will be obtained by minimizing the DBI action with respect to w.

$$S_{\rm DBI} = 4\pi \int d^2x \left(\frac{Z_x^{-1/2} e^{-\Phi}}{(2\pi)^3 \alpha'^2} \sqrt{Z_\Omega w^4 + 4\pi^2 \left(\frac{k\alpha'}{2} - \frac{N\alpha'}{2} \frac{1}{1+\frac{(w-r_0)^2}{\rho_c^2}} \right)^2} \right). \tag{5.30}$$

The tension is minimized at $w \approx r_0$ (in the large N_c limit) and we obtain for the SO(3) symmetric setup

$$T_{\rm D3} \approx 2\pi \frac{m^2}{g_{\rm YM}^2} N_c(N_c - k)\,. \tag{5.31}$$

We will see that the tension of the configuration in figure 4 will be lower than the above and so the SO(3) symmetric k-string cannot be stable.

5.3.2 The k-vortex

Now let us compute the energy of the configuration in figure 4. It consists of two parts: i) one which is a piece of a sphere subtending the solid angle parametrized by

$$0 \le \theta_w \le \bar{\eta}_k\,, \quad \text{and} \quad 0 \le \phi_w \le 2\pi, \tag{5.32}$$

that we refer to as the polar cap, and ii) a disc glued to the bottom of the cap. The two parts are distinguished by the fact that polar cap lies close to the D5 shell at $w \approx r_0$ where the B_2 field reaches its maximum. Since B_2 is non-zero only within a thin region (5.11) of width $\rho_c \ll r_0$, the disc portion of the expanded brane only sees a vanishing antisymmetric tensor potential. In fact the geometry seen by the disc in the interior of the D5-sphere is basically flat. As a consequence of this, the polar cap can minimize its tension by having a magnetic field switched on that completely cancels the pullback of B_2. This means that

the polar cap is close to the D5-sphere and is practically tensionless (suggesting that it is possibly dissolved in the D5). In fact the entire tension of the configuration arises from the disc. The disc itself cannot shrink since its boundary must match on to the boundary of the polar cap, and the size of the latter is fixed by the net D-string charge. Also, in the absence of any B_2 field in the interior of the D5-sphere, the disc has no world-volume magnetic field.

Let us first discuss the polar cap. The D3 world-volume is parameterized by the coordinates $(x_0, x_1, \phi_w, \theta_w)$. With this portion of the brane at constant $|\vec{w}| \approx r_0$ and $\vec{y} = 0$, we take the F_2 field to be proportional to the volume form of the two-sphere

$$F_2 = \frac{k}{(\cos\bar{\eta}_k - 1)} \sin\theta_w (d\theta_w \wedge d\phi_w) \tag{5.33}$$

where $0 \leq \theta_w \leq \bar{\eta}_k$ and $0 \leq \phi_w \leq 2\pi$. The normalization is chosen so that this yields a D-string charge k for the blown up D3-brane. The D1-charge density is concentrated entirely in the polar cap portion.

The DBI action for the probe D3-brane reads

$$S_{\rm cap} = \frac{1}{(2\pi)^3 \alpha'^2} \int d^4\xi \left\{ e^{-\Phi} \sqrt{(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}))} \right\}. \tag{5.34}$$

where

$$G_{ab} + B_{ab} + 2\pi\alpha' F_{ab} = \begin{pmatrix} -Z_x^{-1/2} & 0 & 0 & 0 \\ 0 & Z_x^{-1/2} & 0 & 0 \\ 0 & 0 & Z_\Omega^{1/2} w^2 & (2\pi\alpha' F_{\theta\phi} + B_{\theta\phi}) \\ 0 & 0 & -(2\pi\alpha' F_{\theta\phi} + B_{\theta\phi}) & Z_\Omega^{1/2} w^2 \sin^2\theta_w \end{pmatrix}. \tag{5.35}$$

From this we find the tension of the cap to be

$$S_{\rm cap} = \int d^2x d\phi_w d\theta_w \sin\theta_w \left(\frac{Z_x^{-1/2} e^{-\Phi}}{(2\pi)^3 \alpha'^2} \sqrt{Z_\Omega w^4 + 4\pi^2 \left(\frac{k\alpha'}{1 - \cos\bar{\eta}_k} - \frac{N\alpha'/2}{1 + \frac{(w - r_0)^2}{\rho_c^2}} \right)^2} \right). \tag{5.36}$$

The action for the polar cap needs to be extremized with respect to both w and $\bar{\eta}_k$. We find that there is a minimum at which the tension vanishes exactly for $|\vec{w}| = r_0$ and

$$(1 - \cos\bar{\eta}_k) = \frac{2k}{N_c}. \tag{5.37}$$

Since the action is positive definite and it vanishes exactly at the above values of w and $\bar{\eta}_k$, we have found the global minimum of this contribution to the tension. The pullback of B_2 is exactly cancelled by the magnetic field and the volume of the cycle goes to zero near $w = r_0$ due to vanishing Z_Ω and the brane becomes tensionless. At this point we expect significant stringy corrections and the DBI approach is invalid. These will likely change the tension for the polar cap, but it is not clear whether the result will become comparable to the much larger contribution to the tension from the disc at the bottom of the polar cap.

JHEP12(2008)077

The disc lies for most of its extension at $|\vec{w}| - r_0 \gg \rho_c$. In this limit the relevant metric (at $\vec{y} = 0$) is

$$ds^2\big|_{w-r_0 \gg \rho_c} = \frac{w^2 - r_0^2}{R_{\rm AdS}^2} dx_\mu dx_\nu \eta^{\mu\nu} + \frac{R_{\rm AdS}^2}{w^2 - r_0^2}(dw^2 + w^2(d\theta_w^2 + \sin^2\theta_w d\phi_w^2)) \tag{5.38}$$

and the dilaton is simply $e^\Phi = g_s$. The B_2 field is also small in this region. Hence, the disc portion of the probe D3 has two space-time directions (x_0, x_1), and two directions in the $\vec{w}$ space, without fluxes. The warp factors from the two different subspaces cancel out and then the resulting DBI action is equivalent to the one for a membrane in flat space with fixed perimeter. So the tension of the disc is given by its area in flat space,

$$T_{\rm D3} \approx \frac{1}{(2\pi)^3 \alpha'^2} \frac{1}{g_s} \left(\pi r_0^2 \sin^2\bar{\eta}_k\right) = \frac{m^2}{2g_s} k(N_c - k)\,. \tag{5.39}$$

which is indeed less than the tension of the SO(3) symmetric ansatz in eq. (5.31). This gives the tension of the k-vortex and remarkably, matches our weak coupling semiclassical results and the softly broken $\mathcal{N} = 2^*$ formula. In this analysis we neglected the small region at $w - r_0 \lesssim \rho_c$; a more careful DBI analysis yields

$$T_{\rm D3} = \frac{1}{g_s(2\pi)^2\alpha'^2} \int_0^{r_0 \sin\bar{\eta}_k} \sqrt{\frac{1}{(r_0\cos\bar{\eta}_k)^2 + s^2}\left(s^2 + (r_0\cos\bar{\eta}_k)^2 \frac{(w-r_0)^2}{(w-r_0)^2 + \rho_c^2}\right)}\, sds\,. \tag{5.40}$$

The formula is obtained following a non-trivial cancellation between the dilaton and the metric warp factors. Clearly for $|w - r_0| \gg \rho_c$, the result is given by the area of the flat disc. It is a good approximation to ignore ρ_c relative to the disc radius, since

$$\frac{\rho_c}{r_0} = \sqrt{\frac{g_s}{N_c\pi}} \ll 1. \tag{5.41}$$

It is straightforward to check that a more careful extremization does not change the result at the leading order in g_s.

A rather interesting cross-check of the picture above results when one determines the tension of the same kind of configuration (a polar cap with a disc glued) with generic values of $\bar{\eta}_k$, i.e. where $\bar{\eta}_k$ is allowed to be a free parameter instead of being determined by the k-string charge. The resulting tension formula is then

$$T_{\rm D3}(\bar{\eta}_k, k) = \frac{m^2}{2g_s}\left(\frac{N_c^2 \sin^2\bar{\eta}_k}{4} + N_c \left| N_c \frac{1 - \cos\bar{\eta}_k}{2} - k \right| \right)\,, \tag{5.42}$$

which, indeed always has a minimum at the value of $\bar{\eta}_k$ given by eq. (5.37), as shown in figure 5.

5.3.3 Theta term in world-sheet sigma model

Since our expanded D3-brane configuration breaks the SO(3) isometry to U(1), it follows that small fluctuations of the orientations of the configuration along the D5-sphere, will lead to a $1+1$ dimensional sigma model with target space $\mathbb{CP}^1$. As usual, all potentially

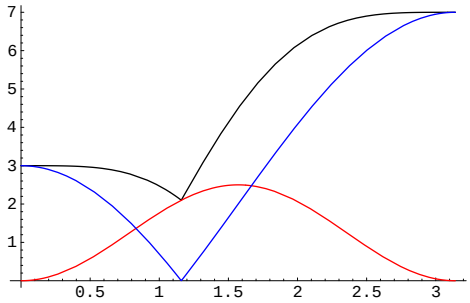

Figure 5: Energy as a function of $\bar{\eta}_k$ for $N = 10$, $k = 3$. The blue line is proportional to the energy of the "polar cap", the red one to the energy of the flat disc and the black line is the sum of the two contributions. The minimum is given by eq. (5.37).

interesting physics lies in the theta angle of this sigma model. From this picture it is straightforward to find also the θ_{1+1} of the effective S^2 sigma model. This comes from the following Chern-Simons coupling in the D3-brane theory,

$$S^{\rm CS}_{\rm cap} = \frac{1}{(2\pi)^3\alpha'^2}\int_{\rm cap} C_2 \wedge (2\pi\alpha' F_2)\,. \tag{5.43}$$

For $\theta_{3+1} \neq 0$, the components of C_2 tangential to the polar cap, can be read off from (5.11) and (5.12). Note that the only non-zero contribution comes from the polar cap, since it has a magnetic field,

$$F_2 = \frac{N_c}{2}\sin\theta_w (d\theta_w \wedge d\phi_w)\,. \tag{5.44}$$

Let us denote with $\vec{n}_w$, the position of the North Pole at the center of the polar cap and for simplicity, let us orient $\vec{n}_w$ in the w^3 direction. We also denote the position vector of any point on the polar cap as

$$\vec{p} = (\sin\theta_w\cos\phi_w, \sin\theta_w\sin\phi_w, \cos\theta_w)\,. \tag{5.45}$$

Now, we want to consider the effect of an infinitesimal displacement of the entire polar cap following from the action of a rotation generator in SO(3)/U(1). We allow this displacement to have an adiabatic dependence on the non-compact coordinates x^0 and x^1. Under the infinitesimal change $\vec{n}_w$ transforms as

$$\vec{n}_w \to \vec{n}_w + \partial_0\vec{n}_w\, dx^0 + \partial_1\vec{n}_w\, dx^1\,. \tag{5.46}$$

The corresponding displacement for a generic point $\vec{p}$ on the polar cap is,

$$\vec{p} \to \vec{p} + \partial_0\vec{p}\, dx^0 + \partial_1\vec{p}\, dx^1\,. \tag{5.47}$$

It is easily seen that the variations in the position vector $\vec{p}$ are related to the change in $\vec{n}_w$ as

$$\partial_s\vec{p} = (\vec{n}_w \times \partial_s\vec{n}_w)\times\vec{p}\,. \tag{5.48}$$

Armed with these relations between the variations in $\vec{p}$ and those in $\vec{n}_w$, we turn to the Chern-Simons term (5.43) in the D-brane action. First of all the pullback of C_2 can be

JHEP12(2008)077

re-expressed in terms of the unit vectors corresponding to the location of each point on the polar cap

$$P[C_2] = -\frac{\theta_{3+1}\alpha' N_c}{2}\ (\partial_0\vec{p}\times\partial_1\vec{p})\cdot\vec{p}\ dx^0\wedge dx^1\,, \tag{5.49}$$

which can be written in terms of the unit polar vector $\vec{n}_w$ as

$$P[C_2] = -\frac{\theta_{3+1}\alpha' N_c}{2}\cos\theta_w\,(\partial_0\vec{n}_w\times\partial_1\vec{n}_w)\cdot\vec{n}_w\,dx^0\wedge dx^1\,. \tag{5.50}$$

We substitute this in eq. (5.43) and integrate over the polar angles of the cap to obtain the topological term in the world-sheet sigma model of the k-vortex

$$\mathcal{L}_\theta = -\frac{k(N_c-k)\,\theta_{3+1}}{8\pi}\,\epsilon^{sr}\vec{n}_w\cdot(\partial_s\vec{n}_w\times\partial_r\vec{n}_w)\,. \tag{5.51}$$

Once again, despite potential issues with regard to the high curvatures in the vicinity of the D5-sphere, the answer is in agreement with the physics at $g^2_{\rm YM}\ll 1$ and is invariant under shifts of the Yang-Mills theta angle by multiples of 2π. As before, one likely reason for the robust nature of the result is that it originates from the Chern-Simons couplings of the D3-brane.

The coefficient of the kinetic term $\tilde{B}_{N_c,k}$ is a trickier issue in the probe D3 approach. We have already remarked that this will flow in the quantized sigma model. Nevertheless it is an object that we can formally estimate. The contribution to this quantity from the cap is zero; there is a non-zero contribution from the disc that we will estimate here. Let us parameterize the disc in the $\vec{w}$ space with the coordinate $\vec{q}$:

$$\vec{q} = (s\cos\phi, s\sin\phi, r_0\cos\bar{\eta}_k)\,. \tag{5.52}$$

The infinitesimal variation of $\vec{q}$ is

$$\frac{\partial\vec{q}}{\partial x^s} = \left(\vec{n}_w\times\frac{\partial\vec{n}_w}{\partial x^s}\right)\times\vec{q}\,, \tag{5.53}$$

which upon inserting in the DBI action, yields

$$\mathcal{L}_{\rm kin} = \tilde{B}_{N_c,k}\left(\frac{\partial\vec{n}_w}{\partial x^s}\right)^2\,, \tag{5.54}$$

where

$$\tilde{B}_{N_c,k} = \int ds d\phi\left(\frac{R^4_{\rm AdS}s\left(s^2\cos^2(\phi)+\cos^2(\bar{\eta}_k)r_0^2\right)\sqrt{\frac{\left((r_0-w)^2+\rho_c^2\right)s^2+\cos^2(\bar{\eta}_k)r_0^2(r_0-w)^2}{\left(s^2+\cos^2(\bar{\eta}_k)r_0^2\right)\left((r_0-w)^2+\rho_c^2\right)}}}{16g_s\pi^3\alpha'^2\,(w+r_0)^2\left((r_0-w)^2+\rho_c^2\right)}\right)\,, \tag{5.55}$$

which scales as $1/\sqrt{g_s}$ and not as $1/g_s$ as we would expect from the field theory and also from the D1 probe calculation. The contribution from the interior of the disc is $\mathcal{O}(g_s^0)$. The leading contribution of order $\mathcal{O}(1/\sqrt{g_s})$ comes from the boundary of the disc at $w-r_0\approx\rho_c$. This is the region nearby the intersection between the cap and the disc and indeed there we can not trust our guess for the shape of the D3 brane probe.

JHEP12(2008)077

5.4 Relation to the baryon vertex

In the context of gravity duals of gauge theories, there exists a close relationship between baryons and flux tubes. Flux tubes are made of the same material as baryons, first seen in the context of the baryon vertex in $\mathcal{N} = 4$ theory [47, 48] and subsequently for gravity duals of confining gauge theories [49 – 51]. In all these examples, a baryon vertex with N_c strings attached, represented by a wrapped D5-brane for example, can be deformed and pulled apart into groups of constituent quarks connected by a flux tube. The portion of the D5-brane that looks like a flux tube in the gauge theory is obtained by the 5-brane wrapping an $S^4 \subset S^5$.

Polchinski and Strassler [8] argued that a D5-brane baryon vertex of the UV $\mathcal{N} = 4$ theory, when taken to the IR by moving towards the interior of the $\mathcal{N} = 1^*$ geometry, eventually meets the dielectric 5-brane spheres in the interior. In the confining vacuum when the baryon vertex is moved past the NS5-sphere, by the Hanany-Witten process of brane creation [52], the D5-brane baryon vertex turns into a D3-brane ball filling the space inside the NS5-sphere. Following a similar logic applied to a D5-brane wrapping an S^4 inside the S^5 in the far UV geometry, in the IR we would expect the $\mathbb{Z}_{N_c}$ flux tube to be a D3-brane with world-volume $\mathbb{R}^{1,1} \times \mathcal{D}_2$ where the $\mathcal{D}_2$ is a disc stretching inside and ending on the dielectric sphere. The magnetic flux tube would essentially be the same type of object, obtained by S-duality on the confining vacuum. It is encouraging to see that the crucial portion of our candidate k-vortex, the expanded D3-brane, is precisely such a D3-brane disc. The tension of the magnetic flux tube arises entirely from this disc. Nevertheless, the polar cap was crucial for providing the boundary condition that stabilized the disc, and for providing the magnetic field responsible for k units of D-string charge. It would be interesting to understand better, the precise connection between the two slightly differing pictures.

5.5 Confining vacuum

We conclude our discussion on flux tubes in the Polchinski-Strassler dual, with a brief analysis of k-strings in the confining vacuum. First of all we note that the tension of the k-string in the confining phase at strong coupling $g^2_{\rm YM} N_c \gg 1$ is simply the S-dual of the magnetic k-string tension in the Higgs vacuum at weak coupling or $N_c/g^2_{\rm YM} \gg 1$. Hence we learn that at least at large N_c, and large 't Hooft coupling, the confining k-strings must obey a Casimir scaling law,

$$T^{\rm confining}_{N_c,k} = m^2 \, \frac{g^2_{\rm YM}}{8\pi} \, k(N_c - k). \tag{5.56}$$

A direct confirmation of this from the corresponding expanded D3-brane in the confining vacuum geometry would be useful, but we leave this for future study. The Casimir scaling for confining string tensions is in contrast to previously encountered sine laws and approximate sine laws in other confining theories [28 – 30]. The confining vacuum is manifestly SO(3) invariant in the absence of any VEVs for the adjoint scalars.

Below we outline the calculation of the tension for a $k = 1$ flux tube in the confining vacuum (first done in [8]) to see how it is consistent with the action of S-duality. The

configuration corresponding to the confining vacuum is an NS5-brane wrapped on a sphere. The corresponding supergravity background can be found via S-duality from the Higgs vacuum with the fields and the parameters transforming as

$$g_s \to \tilde{g}_s = \frac{1}{g_s}, \qquad \alpha' \to \tilde{\alpha}' = g_s \alpha', \qquad \exp(\Phi) \to \exp(\tilde{\Phi}) = \exp(-\Phi), \tag{5.57}$$
$$ds^2 \to d\tilde{s}^2 = g_s \exp(-\Phi) ds^2, \qquad (B_2, C_2) \to (\tilde{B}_2, \tilde{C}_2) = (-C_2, B_2)\,.$$

The profile functions for the new background metric are then

$$Z_x = Z_\Omega = \frac{R^4_{\rm AdS}}{\rho_+^2(\rho_-^2+\rho_c^2)}, \;\; Z_y = \frac{R^4_{\rm AdS}(\rho_-^2+\rho_c^2)}{\rho_+^2\rho_-^4}, \tag{5.58}$$

with the metric having the same form as (5.7). The parameters in the metric and the dilaton are,

$$r_0 = (\alpha' m)\pi g_s N_c, \qquad \rho_c = (\alpha' m)\sqrt{\pi g_s N_c}, \qquad e^{2\Phi} = g_s^2 \frac{\rho_-^2+\rho_c^2}{\rho_-^2}. \tag{5.59}$$

This is the background for $\theta_{3+1} = 0$. The confining string of the gauge theory is identified with the F-string (dissolved in the NS5 sphere), which couples directly to the string metric. The action for the probe fundamental string is

$$S_{\rm F1} = \frac{1}{2\pi\alpha'} \int d^2x \left\{ \sqrt{-\det(G_{ab})} + B_{ab} \right\}, \tag{5.60}$$

with the string oriented in the x_0, x_1 directions. Upon evaluating this action we find

$$\begin{aligned} S_{\rm F1} &= \frac{1}{2\pi\alpha'} \int d^2x (Z_x^{-1/2}) \\ &= \frac{1}{2\pi\alpha'} \int d^2x \frac{\sqrt{y^2+(w+r_0)^2}\sqrt{y^2+(w-r_0)^2+\rho_c^2}}{R^2_{\rm AdS}}. \end{aligned} \tag{5.61}$$

The location of the minimum is the same as in the D5 case and the resulting string tension

$$T_{\rm F1} = \frac{m^2 g_s N_c}{2}, \tag{5.62}$$

is exactly the S-dual of the vortex tension. At the radial position of the string $|w - r_0| \approx \rho_c^2/2r_0 = \alpha' m/2 \ll r_0$, the radius of the sphere with constant w is $\sqrt{\alpha' \pi g_s N_c}$ which is large in string units. It is interesting that the flux tube in the strongly coupled confining vacuum appears to break the global SO(3) invariance, as it is point-like on the sphere. This is counter-intuitive, since the SO(3) is an exact global symmetry of the confining vacuum and not a colour-flavour locked transformation as in the Higgs vacuum. So we do not expect the confining strings to have any orientational zero modes. This should become manifest upon quantizing the associated sigma model, whereby the quantum wavefunction spreads over the entire classical moduli space and the classical zero modes are removed. To obtain the classical sigma model for the flux tube, we can allow the string to fluctuate in the directions tangential to the sphere and these would give the action for the "classical

orientational zero modes" of the confining string flux tube. From the Nambu-Goto action for the string we get

$$S_{\rm F1} = \int d^2x \frac{1}{2\pi\alpha'} \sqrt{{\rm Det} \begin{pmatrix} -Z_x^{-1/2} + (\partial_0 \vec{n}_w)^2 w^2 Z_\Omega^{1/2} & (\partial_0 \vec{n}_w)(\partial_1 \vec{n}_w) w^2 Z_\Omega^{1/2} \\ (\partial_0 \vec{n}_w)(\partial_1 \vec{n}_w) w^2 Z_\Omega^{1/2} & Z_x^{-1/2} + (\partial_1 \vec{n}_w)^2 w^2 Z_\Omega^{1/2}. \end{pmatrix}} \quad (5.63)$$

The effective action at the two-derivative level is

$$S_{\rm F1} = \int d^2x \left(\frac{g_s N_c}{4} (\partial_s \vec{n}_w)^2 \right). \quad (5.64)$$

It is nice to see that the classical coupling is exactly S-dual to the one for the D-string. The theta dependence of the sigma model action could provide us with further clues about the worldsheet dynamics of these flux tubes. However, the confining background above is for $\theta_{3+1} = 0$. Shifting the Yang-Mills theta angle by multiples of 2π changes the vacuum to one in an oblique-confining phase and the NS5 brane to a $(1, n)$ 5-brane. If we dial θ_{3+1} as was done in the Higgs vacuum using an $SL(2, \mathbb{R})$ shift, this does not alter the B_2 field, although it does change the R-R potential C_2. However, the former does not have components along the NS5 sphere, whilst only the RR two form does and we do not know how this couples to the F1 world-sheet in a simple way.

6. Summary and further questions

In this paper we have first studied solitonic k-vortices in the Higgs vacuum of the $SU(N_c)$ $\mathcal{N} = 1^*$ theory with equal adjoint masses, transforming under an $SO(3)_{C+F}$ symmetry group. We have found that for every k and N_c the vortex world-sheet theory is a non-supersymmetric S^2 sigma model. Perhaps the most interesting feature of the two dimensional world-sheet theory is the relation between its theta angle and the four dimensional one, $\theta_{1+1} = k(N_c - k)\theta_{3+1}$. This has very specific implications for the IR dynamics of the sigma model which is asymptotically free and for general values of θ_{1+1}, has a mass gap with the spectrum consisting of a triplet of SO(3). When $\theta_{1+1} = \pi$ however, the theory is integrable and the spectrum consists of massless doublets of SO(3) and the theory flows to a $c = 1$ conformal fixed point. The doublets of SO(3) which are confined into meson-like triplet states become deconfined and massless at $\theta_{1+1} = \pi$. However this value of the world-sheet theta angle corresponds to a seemingly non-special value of the spacetime theta angle. We have speculated on the possibility that the values:

$$\theta_{3+1} = \frac{\pi}{k(N_c - k)},$$

may correspond to a level crossing in the semiclassical spectrum of massive, mutually non-local monopole-dyon states of the parent $\mathcal{N} = 2^*$ theory in the Coulomb phase. These states would be confined in the Higgs vacuum and appear as SO(3) doublets bound to a k-vortex. We have not presented any evidence for this, but clearly, further study of the relation between the vortex world-sheet spectrum and the spectrum of the four dimensional theory will reveal interesting physics.

JHEP12(2008)077

The relation between the theta angles on the worldsheet and spacetime also implies that instantons of charge one in the k-string $\mathbb{CP}^1$ sigma model correspond to multi-instantons in the four dimensional theory.

Yet another feature of our non-Abelian string solutions is that their tensions, evaluated numerically, are extremely well approximated by the Casimir scaling law,

$$T_{N_c,k} = \frac{2\pi m^2 k(N_c - k)}{g^2_{\rm YM}}\,,$$

for large N_c. This is remarkable in that the Casimir scaling formula is known to be valid for vortices in the softly broken $\mathcal{N} = 2^*$ theory because of "almost $\mathcal{N} = 2$ SUSY" and the resulting string solutions are BPS. Our solutions are far from BPS and the agreement with the BPS tension formula, at large N_c is surprising and needs further explanation. The semiclassical Higgs vacuum is mapped by S-duality to the confining vacuum at strong 't Hooft coupling and so confining string tensions in this regime will also obey Casimir scaling.

We have identified the supergravity duals of the vortex strings in the large N_c limit of $\mathcal{N} = 1^*$ theory. The dual IIB string background, due to Polchinski and Strassler, has a parametric regime of validity $N_c/g_s \gg 1$, which includes the semiclassical regime of weak gauge coupling. For this reason, and more explicitly from the form of the metric itself which is sourced by a D5-brane in the IR, it is expected that the IR physics of the Higgs vacuum lies outside the regime of supergravity due to large curvatures. We find it surprising therefore, that we were not only able to identify the candidate objects dual to k-vortices in the large N_c limit, as expanded D3-branes, but also able to compute their tensions and find an exact Casimir scaling in agreement with the semiclassical results. As a bonus we were able to reproduce the semiclassical relation between the Yang-Mills theta angle and the worldsheet theta angle from the candidate wrapped, D-brane configurations. It would definitely be useful to understand better the reason for this agreement and connect to some kind of large N_c BPS property.

The Higgs vacuum of $\mathcal{N} = 1^*$ theory, in the large N_c limit, has been argued to provide a deconstruction of six dimensional supersymmetric gauge theory compactified on a fuzzy sphere [34]. In this picture, the vortices may be reinterpreted as noncommutative instantons [40] of the six-dimensional theory. This presents a potentially fruitful arena for systematically investigating the Higgs vacuum vortices, at least in the large N_c limit [53] , and may explain some of the surprising features above at large N_c. Finally, a closely related situation arises in the beta deformation of $\mathcal{N} = 4$ theory at special values of β, wherein the resulting theory in its Higgs phase has been argued to deconstruct Little String Theory [33].

Acknowledgments

The authors would like to thank A. Armoni, C. Hoyos-Badajoz, K. Konishi, B. Lucini, C. Nuñez and A. Yung for discussions.

A. "Near shell" background: flat D5 and NS5 with D3 charge

Below we quote the form of the "near-shell" background in the Higgs (D5) and confining

(NS5) solutions of Polchinski and Strassler. In the Higgs (confining) phase, the "near-shell" solutions are directly read off from the metric in string frame for p flat D5-branes (NS5 branes) with D3-brane charge bound to them [41, 42]:

D5/D3 background.

$$ds^2 = \frac{dx_\mu\, dx_\nu \eta^{\mu\nu}}{\sqrt{1+\frac{S^2}{\alpha'^2 u^2}}} + \frac{\sqrt{1+\frac{S^2}{\alpha'^2 u^2}}}{\frac{S^2}{\alpha'^2 u^2}\cos^2\varphi + 1}(dx_4^2 + dx_5^2) + \alpha'^2\sqrt{1+\frac{S^2}{\alpha'^2 u^2}}(du^2 + u^2 d\Omega_3^2) \quad \text{(A.1)}$$

where

$$S = \sqrt{\frac{g_s p \alpha'}{\cos\varphi}}\,, \quad \text{(A.2)}$$

and $\tan\varphi$ is proportional to the density of D3 brane charge dissolved on the D5-branes. The dilaton and the B fields are given by:

$$e^{2\Phi} = g_s^2 \frac{\alpha'^2 u^2}{S^2\cos^2\varphi + \alpha'^2 u^2}\,, \qquad B_{45} = -\frac{S^2 \sin\varphi\cos\varphi}{S^2\cos^2\varphi + \alpha'^2 u^2}\,. \quad \text{(A.3)}$$

The RR potentials are:

$$C_2 = \pm 2\frac{S^2\cos\varphi}{g_s}\sin^2\theta\cos\phi_1\, d\theta\wedge d\phi_2\,, \quad \text{(A.4)}$$

$$C_4 = \mp\frac{2S^2\sin\varphi}{g_s}\frac{r^2 + S^2/2\cos^2\varphi}{r^2+S^2\cos^2\varphi}\sin^2\theta\cos\phi_1 dx_5\wedge dx_4\wedge d\theta\wedge d\phi_2 \quad \text{(A.5)}$$
$$\pm\frac{\sin\varphi}{g_s}\frac{r^2}{r^2+S^2}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,,$$

where (θ,ϕ_1,ϕ_2) are the standard coordinates in S^3.

In the decoupling limit $\alpha'\to 0$, $\tan\varphi\to\infty$ with $\alpha'\tan\varphi = b$ held constant, after the rescaling $\tilde{x}_{4,5} = \tan\varphi x_{4,5}$, the solution reads:

$$ds^2 = \alpha'\left(\frac{u}{g_s p a}\left(\frac{d\tilde{x}_4^2 + d\tilde{x}_5^2}{1+a^2u^2} + dx_\mu\, dx_\nu\eta^{\mu\nu}\right) + \frac{g_s p a}{u}(du^2 + u^2 d\Omega_3^2)\right)\,, \quad \text{(A.6)}$$

$$e^{2\Phi} = g_s^2\frac{a^2u^2}{1+a^2u^2}\,, \qquad B_{45} = -\frac{\alpha'}{g_s p a^2}\frac{1}{1+a^2u^2}\,,$$

where we have defined

$$a = \sqrt{\frac{\alpha'\tan\varphi}{g_s p}}\,. \quad \text{(A.7)}$$

The identification used in the Polchinski-Strassler solution to interpolate between the near-shell and the asymptotic metric is,

$$u = \frac{\rho_-}{\alpha'}, \qquad a = \frac{1}{p\, m\sqrt{g_s N_c \pi}} \qquad \tilde{x}^{4,5} = \frac{1}{p\,\alpha' m^2 \pi N_c} w^{1,2}\,. \quad \text{(A.8)}$$

NS5/D3 background. The metric in string frame for p flat NS5-branes with D3-brane charge (in the appropriate decoupling limit) is,

$$ds^2 = \frac{\alpha' {g_s}^2}{pa^2}\left(\frac{d\tilde{x}_4^2 + d\tilde{x}_5^2}{\sqrt{1+a^2u^2}} + \sqrt{1+a^2u^2}dx_\mu dx_\nu \eta^{\mu\nu}\right) + \alpha' p\frac{\sqrt{1+a^2u^2}}{u^2}(du^2 + u^2 d\Omega_3^2)\,. \quad \text{(A.9)}$$

The dilaton and the C_2 fields are:

$$e^{2\tilde{\Phi}} = {g_s}^2\frac{1+a^2u^2}{a^2u^2}, \qquad C_2 = -\frac{{g_s}^2\alpha'}{a^2 p}\frac{1}{1+a^2u^2}d\tilde{x}_4 \wedge d\tilde{x}_5\,. \quad \text{(A.10)}$$

In the confining vacuum, the identification used in the PS solution between near-shell and the asymptotic metric is,

$$u = \frac{\rho_-}{\alpha' g_s}, \qquad a = \frac{\sqrt{g_s}}{p\,m\sqrt{N_c\pi}} \qquad \tilde{x}^{4,5} = \frac{1}{p\,g_s\alpha' m^2\pi N_c}w^{1,2}\,. \quad \text{(A.11)}$$

References

[1] N. Seiberg and E. Witten, *Monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485] [hep-th/9407087]; *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[2] A. Abrikosov, *On the magnetic properties of superconductors of the second group*, *Sov. Phys. JETP* **5** (1957) 1174 [*Zh. Eksp. Teor. Fiz.* **32** (1957) 1442].

[3] H.B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[4] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[5] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[6] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *JHEP* **07** (1999) 021 [hep-th/9906011].

[7] N. Dorey and S.P. Kumar, *Softly-broken $N = 4$ supersymmetry in the large-N limit*, *JHEP* **02** (2000) 006 [hep-th/0001103].

[8] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[9] M.J. Strassler, *Messages for QCD from the superworld*, *Prog. Theor. Phys. Suppl.* **131** (1998) 439 [hep-lat/9803009]; *Millennial messages for QCD from the superworld and from the string*, hep-th/0309140.

[10] C. Montonen and D.I. Olive, *Magnetic monopoles as gauge particles?*, *Phys. Lett.* **B 72** (1977) 117.

[11] A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [hep-th/0306150].

[12] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nonabelian superconductors: vortices and confinement in $N = 2$ SQCD*, *Nucl. Phys.* **B 673** (2003) 187 [hep-th/0307287].

JHEP12(2008)077

[13] D. Tong, *Monopoles in the Higgs phase*, *Phys. Rev.* **D 69** (2004) 065003 [hep-th/0307302].

[14] M. Shifman and A. Yung, *Non-Abelian string junctions as confined monopoles*, *Phys. Rev.* **D 70** (2004) 045004 [hep-th/0403149].

[15] A. Hanany and D. Tong, *Vortex strings and four-dimensional gauge dynamics*, *JHEP* **04** (2004) 066 [hep-th/0403158].

[16] M. Shifman and A. Yung, *Supersymmetric solitons and how they help us understand non-Abelian gauge theories*, *Rev. Mod. Phys.* **79** (2007) 1139 [hep-th/0703267].

[17] D. Tong, *TASI lectures on solitons*, hep-th/0509216.

[18] D. Tong, *Quantum vortex strings: a review*, arXiv:0809.5060.

[19] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Solitons in the Higgs phase: the moduli matrix approach*, *J. Phys.* **A 39** (2006) R315 [hep-th/0602170].

[20] V. Markov, A. Marshakov and A. Yung, *Non-Abelian vortices in $N = 1^*$ gauge theory*, *Nucl. Phys.* **B 709** (2005) 267 [hep-th/0408235].

[21] H.J. de Vega and F.A. Schaposnik, *Electrically charged vortices in nonabelian gauge theories with Chern-Simons term*, *Phys. Rev. Lett.* **56** (1986) 2564; *Vortices and electrically charged vortices in nonAbelian gauge theories*, *Phys. Rev.* **D 34** (1986) 3206;
J. Heo and T. Vachaspati, *Z_3 strings and their interactions*, *Phys. Rev.* **D 58** (1998) 065011 [hep-ph/9801455];
F.A. Schaposnik and P. Suranyi, *New vortex solution in* SU(3) *gauge-Higgs theory*, *Phys. Rev.* **D 62** (2000) 125002 [hep-th/0005109];
K. Konishi and L. Spanu, *Non-Abelian vortex and confinement*, *Int. J. Mod. Phys.* **A 18** (2003) 249 [hep-th/0106175].

[22] M.A.C. Kneipp, *BPS Z_N string tensions, Sine and Casimir laws and integrable field theories*, *Phys. Rev.* **D 76** (2007) 125010 [arXiv:0707.3791]; *Color superconductivity, Z_N flux tubes and monopole confinement in deformed $N = 2^*$ super Yang-Mills theories*, *Phys. Rev.* **D 69** (2004) 045007 [hep-th/0308086].

[23] E. Witten, *Instantons, the quark model and the $1/N$ expansion*, *Nucl. Phys.* **B 149** (1979) 285.

[24] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253.

[25] A.B. Zamolodchikov and A.B. Zamolodchikov, *Massless factorized scattering and σ-models with topological terms*, *Nucl. Phys.* **B 379** (1992) 602.

[26] V.A. Fateev, E. Onofri and A.B. Zamolodchikov, *The Sausage model (integrable deformations of $O(3)$ σ-model)*, *Nucl. Phys.* **B 406** (1993) 521.

[27] B. Lucini and M. Teper, *Confining strings in* SU(N) *gauge theories*, *Phys. Rev.* **D 64** (2001) 105019 [hep-lat/0107007];
B. Lucini, M. Teper and U. Wenger, *Glueballs and k-strings in* SU(N) *gauge theories: calculations with improved operators*, *JHEP* **06** (2004) 012 [hep-lat/0404008].

[28] C.P. Herzog and I.R. Klebanov, *On string tensions in supersymmetric* SU(M) *gauge theory*, *Phys. Lett.* **B 526** (2002) 388 [hep-th/0111078].

JHEP12(2008)077

[29] M.R. Douglas and S.H. Shenker, *Dynamics of* SU(N) *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 271 [hep-th/9503163].

[30] A. Hanany, M.J. Strassler and A. Zaffaroni, *Confinement and strings in MQCD*, *Nucl. Phys.* **B 513** (1998) 87 [hep-th/9707244].

[31] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053].

[32] A. Gorsky, M. Shifman and A. Yung, *Non-Abelian Meissner effect in Yang-Mills theories at weak coupling*, *Phys. Rev.* **D 71** (2005) 045010 [hep-th/0412082].

[33] N. Dorey, *S-duality, deconstruction and confinement for a marginal deformation of* $N = 4$ *SUSY Yang-Mills*, *JHEP* **08** (2004) 043 [hep-th/0310117]; *A new deconstruction of little string theory*, *JHEP* **07** (2004) 016 [hep-th/0406104].

[34] R.P. Andrews and N. Dorey, *Spherical deconstruction*, *Phys. Lett.* **B 631** (2005) 74 [hep-th/0505107]; *Deconstruction of the Maldacena-Núñez compactification*, *Nucl. Phys.* **B 751** (2006) 304 [hep-th/0601098].

[35] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[36] M. Edalati and D. Tong, *Heterotic vortex strings*, *JHEP* **05** (2007) 005 [hep-th/0703045]; D. Tong, *The quantum dynamics of heterotic vortex strings*, *JHEP* **09** (2007) 022 [hep-th/0703235];
M. Shifman and A. Yung, *Heterotic flux tubes in* $N = 2$ *SQCD with* $N = 1$ *preserving deformations*, *Phys. Rev.* **D 77** (2008) 125016 [arXiv:0803.0158]; *Large-*N *solution of the heterotic* $N = (0, 2)$ *two-dimensional* $CP(N-1)$ *model*, *Phys. Rev.* **D 77** (2008) 125017 [arXiv:0803.0698].

[37] I. Affleck, *Exact critical exponents for quantum spin chains, nonlinear* σ*-models at* $\Theta = \pi$ *and the quantum Hall effect*, *Nucl. Phys.* **B 265** (1986) 409;
I. Affleck and F.D.M. Haldane, *Critical theory of quantum spin chains*, *Phys. Rev.* **B 36** (1987) 5291;
I. Affleck, *Nonlinear* σ*-model at* $\Theta = \pi$*: Euclidean lattice formulation and solid-on-solid models*, *Phys. Rev. Lett.* **66** (1991) 2429.

[38] D. Controzzi and G. Mussardo, *On the mass spectrum of the two-dimensional* $O(3)$ σ*-model with theta term*, *Phys. Rev. Lett.* **92** (2004) 021601 [hep-th/0307143].

[39] R. Shankar and N. Read, *The* $\Theta = \pi$ *nonlinear* σ*-model is massless*, *Nucl. Phys.* **B 336** (1990) 457.

[40] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative* R^4 *and* $(2, 0)$ *superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[41] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-*N *noncommutative field theories*, *JHEP* **11** (1999) 007 [hep-th/9909215].

[42] J.C. Breckenridge, G. Michaud and R.C. Myers, *More D-brane bound states*, *Phys. Rev.* **D 55** (1997) 6438 [hep-th/9611174].

[43] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *JHEP* **05** (2000) 048 [hep-th/0003037];
W. Taylor, *D2-branes in B fields*, *JHEP* **07** (2000) 039 [hep-th/0004141];
A. Alekseev, A. Mironov and A. Morozov, *On B-independence of RR charges*, *Phys. Lett.* **B 532** (2002) 350 [hep-th/0005244];
J.-G. Zhou, *D-branes in B fields*, *Nucl. Phys.* **B 607** (2001) 237 [hep-th/0102178].

JHEP12(2008)077

[44] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109].

[45] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [hep-th/0603208].

[46] S.A. Hartnoll and S. Prem Kumar, *Multiply wound Polyakov loops at strong coupling*, *Phys. Rev.* **D 74** (2006) 026001 [hep-th/0603190].

[47] C.G. Callan and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147].

[48] E. Witten, *Baryons and branes in anti de Sitter space*, *JHEP* **07** (1998) 006 [hep-th/9805112].

[49] C.G. Callan Jr., A. Guijosa and K.G. Savvidy, *Baryons and string creation from the fivebrane worldvolume action*, *Nucl. Phys.* **B 547** (1999) 127 [hep-th/9810092].

[50] C.G. Callan Jr., A. Guijosa, K.G. Savvidy and O. Tafjord, *Baryons and flux tubes in confining gauge theories from brane actions*, *Nucl. Phys.* **B 555** (1999) 183 [hep-th/9902197].

[51] S.A. Hartnoll and R. Portugues, *Deforming baryons into confining strings*, *Phys. Rev.* **D 70** (2004) 066007 [hep-th/0405214].

[52] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[53] R. Auzzi and S.P. Kumar, work in progress.

JHEP12(2008)077